# From distances and operator norms to normed categories

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For the study of normed vector spaces or of similar structures one may not like to be confined from the outset to the consideration of morphisms that respect the algebraic and metric structure of the objects of interest as smoothly as the 1-Lipschitz linear operators do. At least initially it may be preferable to work in a categorical environment with less perfectly behaved morphisms, but that instead comes with an operator norm which, amongst all linear maps, will *then* help to pinpoint classes of topologically or metrically interesting morphisms, such as the bounded or the 1-Lipschitz operators.

Having such general categorical environment, the first questions we may want to ask are: What does Cauchy convergence mean? What is completeness with respect to such a notion of convergence, and do completions exist? Is there a "mother category" which, in normed category theory, may play the role that Set plays in ordinary category theory? Are there protagonist normed categories, and is the category of normed vector spaces and linear operators amongst them? Are there unexpected examples, and how do the notions designed for large categories fare when applied to an individual space considered as a small normed category, like a metric space?

In this talk we will try to address all of these questions. While we base our approach on the notion of normed category as suggested by Lawvere [1], unlike several later articles on the subject we do not amend his axioms but follow rather strictly the original design of the normed structure as an enrichment. Indeed, the guidance of enriched category theory has helped us finding answers to some of the questions posed. Although we allow norms to be quantale-valued, rather than just real-valued, and thereby substantially increase the range of potential applications, the talk should remain understandable for an audience with basic knowledge of category theory, including that of adjunctions.

This presentation was preceded by the talks [3, 4]. Other than to the original work [1], we also refer to the article [2] which has influenced our choice of examples. For further literature and all mathematical details not covered in the talk, we refer to the paper [5].

## References

[1] F.W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. Fis. Milano 43, 135–166, 1973. Republished in: Reprints in Theory Appl. Categories 1, 2002. https://www.emis.de/journals/TAC/reprints/articles/1/tr1.pdf .

[2] W. Kubiś, *Categories with norms*, 2017. https://arxiv.org/pdf/1705.10189 .

[3] W. Tholen, *Quantale-weighted Categories*, Talk given at: 106th Peripatetic Seminar on Sheaves and Logic, Brno, May 2022. http://www.math.muni.cz/ bourkej/PSSL106.html .

[4] D. Hofmann, *Cauchy completeness for normed categories*, Talk given at: Coimbra Category Seminar, October 2023. https://www.mat.uc.pt/ categ/pcs14/slides/Hofmann.pdf .

[5] M.M. Clementino, D. Hofmann and W. Tholen, *Cauchy convergence in $\mathcal{V}$-normed categories*, 2024. https://arxiv.org/abs/2404.09032 .

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