Proof Theory, Logic and Algebra

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- How to check the existence of a proof?
- How to check the equality of proofs?

The Semantics of Proofs

Positive cases when we have a proof or two proofs are equivalent is not that hard as we must come up with a construction. (Although, it would be better if we can avoid the syntax altogether). However, when it comes to the negative ones, the problems can be really hard.

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The Semantics of Proofs

Think about the world of all bicartesian closed categories as the models for the abstract proofs. The free category lives high above and the Heyting algebras (preorder bicartesian closed categories) live at the bottom and there are many categories in between. Any positive claim (existence of a proof, equality of proofs, etc) is inherited from the free category to all the shadows. Hence, if you want to prove something negative, it is enough to prove it for a shadow category, where hopefully it is concrete.

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Strong Completeness

Of course checking the positive statements in all bicartesain closed categories is enough as the free category is one of them. However, it is not informative. Therefore, to have a useful completeness, it is reasonable to find some or if we are lucky just one bicartesian closed category that is enough to check.

The Realizations of the Free Category

Let $\mathcal{C}$ be a bicartesian closed category. Then, a function I mapping the formulas of the language $\{\top, \bot, \wedge, \vee, \to\}$ into the objects of $\mathcal{C}$ (and the natural deduction proof of to the morphisms of $\mathcal{C}$) is a formula interpretation (interpretation), if:

- $I(\top) = 1$ and $I(\bot) = 0$,
- $I(A \wedge B) = I(A) \times I(B)$ and $I(A \vee B) = I(A) + I(B)$,
- $I(A \to B) = I(B)^{I(A)}$,

The Realizations of the Free Category

- the axiom cases goes to the identity,
- the juxtaposition to the composition,
- the unique proof of $\top$ from A goes to $! : I(A) \to 1$,
- the unique proof of A from $\bot$ goes to $! : 0 \to I(A)$,
- the application of the rule $\wedge I$ over π and σ goes to $\langle I(\pi), I(\sigma) \rangle$,
- the application of the rule $\wedge E_1$ over π goes to $p_0 I(\pi)$,
- the application of the rule $\wedge E_2$ over π goes to $p_1 I(\pi)$,
- the application of the rule $\vee I_1$ over π goes to $i_0 I(\pi)$,
- the application of the rule $\vee I_1$ over π goes to $i_1 I(\pi)$,
- the application of the rule $\vee E$ over π and σ goes to $[I(\pi)), I(\sigma)]$,
- ...

The Realizations in Practice

Theorem

Let $\mathcal{C}$ be a bicartesian closed category and v is an assignment of the objects of $\mathcal{C}$ to the atoms $\{p_0, p_1, \cdots\}$. Then, there is a unique interpretation extending v.

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Now, we focus on the two problems. The existence of proofs and the equality of proofs.

The Existence of Proofs

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- Is there a proof for $p \vee \neg p$? No! Because, if there is, then there is a map $1 \to I(p) + 0^{I(p)}$, for any bicartesian closed category and any formula-interpretation I. As only the existence of maps is important, we use preorders. Use for instance the frame $\mathcal{O}(\mathbb{R})$ and set $I(p) = (0, \infty)$. Then, $0^{I(p)} = (-\infty, 0)$ and hence $I(p) + 0^{I(p)} = \mathbb{R} - \{0\}$. Therefore, there is no map in $1 \to I(p) + 0^{I(p)}$ as $\mathbb{R} \nsubseteq \mathbb{R} - \{0\}$.

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- Is it possible to come up with a more proof theoretical BHK-type counter-example?

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- Is it possible to come up with a more proof theoretical BHK-type counter-example? Use $\mathbf{Set}^{\mathbb{Z}}$ and set $I(p) = (\{0, 1\}, \sigma)$, where $\sigma(0) = 1$ and $\sigma(1) = 0$. Note that $0^{I(p)}$ is empty, as there is no function from $\{0, 1\}$ to $\varnothing$. Therefore, if there is a map from $1 \to I(p) + 0^{I(p)}$, then as $1 = (\{*\}, id_{\{*\}})$, it means that there is an invariant element in $I(p) + 0^{I(p)}$. As there is no element in $0^{I(p)}$, there must be an invariant element in $\{0, 1\}$, which is not possible.

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equivalent? No! If they are equivalent, then all of their interpretations must be equal. Use **Set** and set $I(p) = \{0, 1\}$. Then, the interpretation of the proofs are $p_0 : \{0, 1\} \times \{0, 1\} \to \{0, 1\}$ and $p_1 : \{0, 1\} \times \{0, 1\} \to \{0, 1\}$. These two functions are not equal.

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- Are the statements p and $p \wedge p$ equivalent? No! Use **Set** and set $I(p) = \{0,1\}$. Then, p and $p \wedge p$ are $\{0,1\} \times \{0,1\}$ and $\{0,1\}$, respectively. If there is an isomorphism between the statements, then there must be an isomorphism between $\{0,1\} \times \{0,1\}$ and $\{0,1\}$, which is not the case.

Strong Completeness for the Existence Problem

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Theorem (McKinsey-Tarski)

$\mathsf{IPC} \vdash A \to B$ iff $I(A) \subseteq I(B)$, for any formula-interpretation $I : \mathcal{L}_{\mathsf{IPC}} \to \mathcal{O}(\mathbb{R})$.

What is the logic of **Set**?

Is it possible to find a concrete non-preordered category with the completeness property? It would be good if it is a **Set**-like category as it is easy to work with and also it captures the BHK interpretation in a more faithful manner.

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Proof.

As the classical logic is a maximal consistent logic, it is just enough to show the existence of a map $\{*\} \to I(A) + \varnothing^{I(A)}$ or equivalently that $I(A) + \varnothing^{I(A)}$ is non-empty. But, this is clear, because either A is non-empty and so is $I(A) + \varnothing^{I(A)}$ or A is empty and hence there is a function in $I(A) \to \varnothing$ which means $\varnothing^{I(A)}$ is non-empty and so is $I(A) + \varnothing^{I(A)}$.

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What is the logic of $\mathbf{Set}^{\mathbb{Z}}$?

Let I be a formula-interpretation $I : \mathcal{L}_{\mathsf{IPC}} \to \mathbf{Set}^{\mathbb{Z}}$. Then, I validates the formula $\neg p \vee \neg\neg p$. To prove that, it is enough to show that for any object (A, σ_A) in $\mathbf{Set}^{\mathbb{Z}}$, either there is an invariant element in $\neg A$ or an invariant element in $\neg\neg A$. If A is empty, then $id_{\varnothing} \in \neg A$ and it is clearly invariant. Otherwise, A is non-empty and hence $\neg A$ must be empty. Therefore, $id_{\varnothing} \in \neg\neg A$ and it is clearly invariant.

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Note that it is impossible to use the same argument to show the validity of $A \vee \neg A$, because when A is non-empty, there is no reason to have an invariant element in A. Recall the example we had before.

A Concrete BHK Interpretation

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Läuchli Realizability

There exists a weak interpretation $I : \mathcal{L}_{\mathrm{IPC}} \to \mathbf{Set}^{\mathbb{Z}}$ such that $\mathrm{IPC} \vdash A \to B$ iff there exists $f : I(A) \to I(B)$ in $\mathbf{Set}^{\mathbb{Z}}$.

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You may think about the possibility of completeness of IPC for weak interpretations in **Set**. That is not possible, as in the classical logic the Pierce law $(((p \to q) \to p) \to p)$ is valid and hence **Set** validates it. But this is not provable in IPC and also it does not include $\bot$ and hence changing the meaning of $\bot$ cannot help it.

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Proof.

First, observe that $A \times 0 \cong 0$. The reason is $Hom(A \times 0, B) \cong Hom(0, B^A)$ and hence there is exactly one map from $A \times 0$ to B, for any B. Now, let $f : A \to 0$ be a map. Then, consider the maps $\langle id_A, f \rangle : A \to A \times 0$ and $p_0 : A \times 0 \to A$ and note that one of the compositions is id_A. The other composition is from $0 \times A$ to itself and as it is a zero object, this map must be the identity $id_{A \times 0}$. Hence, $\langle id_A, f \rangle$ is an isomorphism. Therefore, A is a zero object and hence f is unique as a map from A to 0. The second part is a consequence of $Hom(B, \neg C) \cong Hom(B \times C, 0)$. □

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- For some, this is counter-intuitive, as the negative formulas can have different proofs. For them, $\bot$ is not the initial object. Sometime it is the weak initial object, sometimes they change the logic to the minimal logic, getting rid of $\bot$, altogether.
- For the others, however, even that bold statement is justifiable at some level, as all the proofs of a negative formula are at some level equivalent. The reason is mostly the known fact that the negative statements carry no information and hence their proofs must be unique and even trivial.

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Based on the theorem we saw, the middle ground and the strongest case are equivalent. In the weakest sense, there are many models for the classical proofs such as **Set**. However, the weakest sense just ensures the existence of a proof without identifying its relation to the other proofs. This cannot be reasonable. Observing the classical mentality, it is reasonable to assume that for a classical mathematician, the formulas A and $\neg\neg A$ are equivalent.

Joyal's Lemma

In a bicartesian closed category $\mathcal{C}$, if $A \cong \neg\neg A$, for any object A, then $\mathcal{C}$ is a preorder. If it is a poset as well, it is nothing but a boolean algebra.

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Proof.

By the previous lemma, there is at most one map in $Hom(A, \neg\neg B)$. As $B \cong \neg\neg B$, there is at most one map in $Hom(A, B)$. $\square$

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Philosophically speaking, this implies that in the strongest sense, there is no non-trivial classical proof. This provides a formal explanation that why the classical mathematicians do not care about proofs.

The Strong Completeness for the Equality of Proofs

Similarly, we know that we must investigate the equality in some concrete bicartesian closed categories. But which ones? Think about the free algebras. Working with them is not easy as they are syntactical. For instance, when we have two elements in a free algebra, it is useful to have a way to compute them in some concrete algebra to show if they are equivalent.

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For instance, let $\langle X \rangle$ be the free abelian group generated by a the set X. We know that it is enough to check if two elements are equal in $\langle X \rangle$ by seeing their values under all possible homomorphisms to $(\mathbb{Z}, +)$.

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Theorem (Friedman, Simpson)

For any two natural deduction proofs π and σ of B from the assumptions $A_1, \cdots, A_n$, they are $\beta\eta$ equivalent iff for any interpretation $I : \mathcal{L}_{\text{IPC}} \to \mathbf{Set}$, we have $I(\pi) = I(\sigma) : I(A_1) \times \cdots \times I(A_n) \to I(B)$.

- The function symbols of the language are $\{0, s, +, \cdot\}$.
- We assume that the language does not have $\vee$. Later, we will see that $\exists z[(z = 0 \to A) \wedge (z \neq 0 \to B)]$ can play the role of $A \vee B$.

Then, define Heyting (Peano) arithmetic as the theory consisting of intuitionistic (classical) logic together with the axioms:

- $\forall x(s(x) \neq 0)$,
- $\forall xy(s(x) = s(y) \to x = y)$,
- $\forall x(x + 0 = x)$,
- $\forall xy(x + s(y) = s(x + y))$,
- $\forall x(x \cdot 0 = 0)$,
- $\forall xy(x \cdot s(y) = x \cdot y + x)$,

and the induction scheme $A(0) \wedge \forall x(A(x) \to A(s(x))) \to \forall x A(x)$, for any formula $A(x)$.

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Is it possible to prove the consistency of arithmetic? You may object that using the usual model of arithmetic, we know that the theory is consistent and we can intuitively count on this model. It is 2022, not 1922! Right? OK! But what if I come up with some rather alternative theories describing some hypothetical alternative worlds and what if the theories are inconsistent with the classical model of natural numbers. Then, proving consistency is actually proving the possibility of such a hypothetical world that is orthogonal to our world.

The Russian Constructivism

- For instance, think about the arithmetical theory $\mathrm{HA} + CT$, where CT is the Church thesis:

$$\mathrm{HA} + \forall x \exists y A(x, y) \to \exists e \forall x A(x, e \cdot x)$$

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stating that if you have a total relation, then you can come up with a computable function witnessing that. The theory is describing the alternative constructive world, where every construction is computable. This is a fragment of what is called the Russian arithmetic.

The Russian Constructivism

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Therefore, if CT is true in the standard model, then we have

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- How to prove the consistency of $\mathrm{HA} + CT$? Isn't the recursive world of constructions, **Rec**, useful?

Thank you for your attention!