Equidivisibility and profinite coproduct

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Part I

Background

Equidivisible semigroups

For a semigroup S, let S^I be the monoid obtained from S by adjoining a neutral element I.

A semigroup S is equidivisible if every factorization

$$uv = xy$$

has a common refinement, that is, there is $t \in S^I$ such that

$$\begin{cases} ut = x \\ v = ty \end{cases} \quad \text{or} \quad \begin{cases} u = xt \\ tv = y \end{cases}$$

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Examples

- Free semigroups
- Groups

Theorem (McKnight & Storey, 1969)

A semigroup S is completely simple if and only if for every factorization

$$uv = xy$$

there are $t_1, t_2 \in S^I$ such that

$$\begin{cases} ut_1 = x \\ v = t_1 y \end{cases} \quad \underline{\textbf{and}} \quad \begin{cases} u = xt_2 \\ t_2 v = y \end{cases}$$

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A positive fact

Coproducts in the category of semigroups: free products.

Theorem (McKnight & Storey, 1969)

For every nonempty family $(S_i)_{i \in I}$ of equidivisible semigroups, its free product

$$\mathop{\ast}_{i \in I} S_i$$

is equidivisible.

Pseudovarieties

A pseudovariety of semigroups is a class V of finite semigroups such that

$$\mathsf{V} = \mathbb{H}(\mathsf{V}) = \mathbb{S}(\mathsf{V}) = \mathbb{P}_{\mathsf{fin}}(\mathsf{V})$$

Examples

- Sgp — finite semigroups
- Grp — finite groups
- Ap — finite aperiodic semigroups
- CS — finite completely simple semigroups
- J — finite aperiodic semigroups satisfying $(xy)^\omega = (yx)^\omega$
  where s^ω denotes the unique idempotent power of s
- DS — largest pseudovariety not containing B_2

Relatively free profinite semigroups

- Pro-V semigroups are inverse limits of semigroups of V (in the category of compact semigroups)
- The A-generated free pro-V semigroup $\overline{\Omega}_A \mathsf{V}$ exists.
- $\widehat{A^+} \cong \overline{\Omega}_A \mathsf{Sgp}$

Theorem (†)

For every finite set A,
the free profinite semigroup $\widehat{A^+}$ is equidivisible.

More generally, $\overline{\Omega}_A \mathsf{V}$ is equidivisible if

- $\mathsf{V} \supseteq \mathsf{Ap}$
- $\overline{\Omega}_A \mathsf{V}$ *has open multiplication*

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Almeida & C (2009)
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Part II

When are all free pro-V semigroups equidivisible?

Equidivisibility makes a "pseudoword" a **linearly ordered** labeled set, so that a pseudoword feels more like an ordinary word.

- J. Almeida, A. Costa, J.C. Costa and M. Zeitoun,
  The linear nature of pseudowords, Publicacions Matemàtiques 63 (2019)
- S.J. van Gool, B. Steinberg,
  Pro-aperiodic monoids via saturated models. Isr. J. Math. 234, (2019)
- Works by A. Moura, and by M. Kufleitner and his co-authors, on the pseudovariety $\mathsf{DA} := \mathsf{DS} \cap \mathsf{Ap}$
  - Antecedent work by Almeida & Zeitoun (& J.C. Costa) on the pseudovariety R, the largest where the prefix quasi-order of pseudowords is a partial order.
- It may provide inspiration for when we no longer have equidivisibility, e.g. A. Costa & A. Escada, **Bases for pseudovarieties closed under bideterministic product**, Algebra Universalis 80 (2019)

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The two-sided Cayley graph

The two-sided Cayley graph of an onto homomorphism $\varphi\colon A^+ \to S$ is what one expects:

- the vertices are the elements of $S^I \times S^I$
- we have an edge, labeled by the letter a, from (x, y) to (x', y') if

$$x\varphi(a) = x' \qquad y = \varphi(a)y'$$

Note that:

- we have a path, labeled by the word $u \in A^+$, from (x, y) to (x', y'), if

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Two-sided Karnofsky–Rhodes expansion

Consider an onto homomorphism

$$\varphi\colon A^+ \to S$$

For every $u \in A^+$, the associated path is the path

$$\Big(I, \varphi(u)\Big) \xrightarrow{u} \Big(\varphi(u), I\Big)$$

We denote by $\tau(u)$ the set of transition edges (i.e., not inside a strongly connected component of the graph) of this path.

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We have the following congruence $\equiv_\varphi$ on A^+:

$$u \equiv_\varphi v \Leftrightarrow \begin{cases} \varphi(u) = \varphi(v) \\ \tau(u) = \tau(v) \end{cases}$$

The two-sided Karnofsky–Rhodes expansion of S (by φ) is

$$S_\varphi^{\mathrm{KR}} = A^+/{\equiv_\varphi}$$

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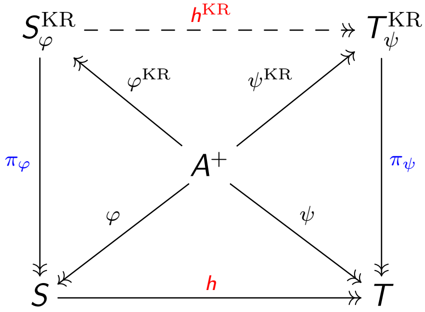
Two-sided Karnofsky–Rhodes expansion

In semigroup theory, an expansion
is an endofunctor F in the category **Semigroups**
together with a natural transformation $F \Rightarrow Id_{\textbf{Semigroups}}$



Two-sided Karnofsky–Rhodes expansion

In semigroup theory, an expansion cut to generators
is an endofunctor F in the category **Semigroups_generated_by_A**
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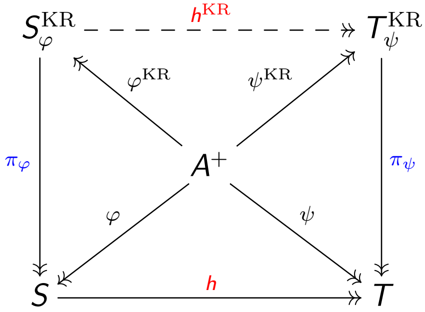
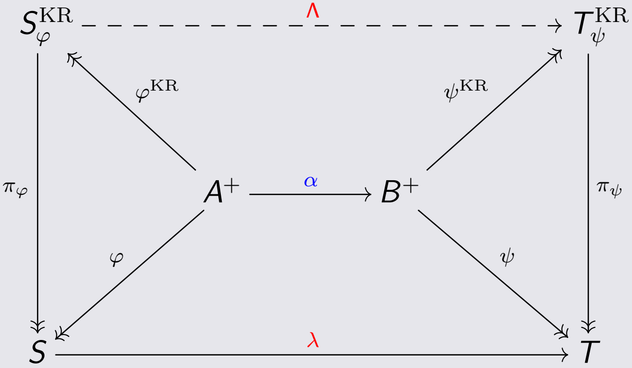


Two-sided Karnofsky–Rhodes expansion

Theorem (Mário Branco, 2006)



Closure under the expansion

A class $\mathcal{C}$ is closed under two-sided Karnofsky–Rhodes expansion when

$$S \in \mathcal{C} \Rightarrow S_{\varphi}^{\mathrm{KR}} \in \mathcal{C}$$

Examples

- the pseudovariety of all finite semigroups
- DA
- DS
- the complexity pseudovarieties

$$\mathsf{C}_n = \underbrace{\mathsf{Ap} * \mathsf{Grp} * \mathsf{Ap} * \cdots * \mathsf{Ap} * \mathsf{Grp} * \mathsf{Ap}}_{\text{semidirect product with } n \text{ occurences of } \mathsf{Grp}}$$

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Equidivisible relatively free profinite semigroups

Theorem (Almeida & C, 2017)

The following are equivalent:

- *all finitely generated free pro-*V *semigroups are equidivisible*
- V *is closed under two-sided Karnofsky–Rhodes expansion or* $\mathsf{V} \subseteq \mathsf{CS}$

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A cancellation property

Let S be a profinite semigroup generated by a finite subset A.

Suppose:

For every $a, b \in A$ and $u, v \in S$,

$$au = bv \Rightarrow \begin{cases} a = b \\ u = v \end{cases}$$

and

$$ua = vb \Rightarrow \begin{cases} a = b \\ u = v \end{cases}$$

We call a semigroup with this property

letter super-cancellative

If V is closed under two-sided Karnofsky–Rhodes expansion, then all $\overline{\Omega}_A \mathsf{V}$, with A finite, are

- equidivisible
- letter super-cancellative

Part III

Profinite coproducts of equidivisible profinite semigroups

Profinite coproduct

Every nonempty family $(S_i)_{i \in I}$ of pro-V semigroups has a coproduct,

$$\coprod_{i \in I}^{\mathsf{V}} S_i$$

dubbed the V-coproduct.

Embedding of the free product

Theorem

The natural mapping

$$\mathop{\ast}_{i\in I} S_i \to \coprod_{i\in I}^{\mathbf{V}} S_i$$

has dense image.

It is injective under a mild condition:

$$\mathbf{V} = \mathbf{N} ⓜ \mathbf{V}$$

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KR-cover

Def. The profinite semigroup S is a KR-cover of T when

- $|T| < \infty$
- $\varphi(S) = T$ for some continuous homomorphism φ
- for every such φ,

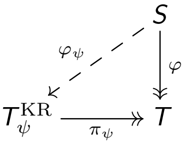
  $\exists$ generating mapping

$$\psi : A \to T \qquad |A| < \infty$$

  $\exists$ continuous homomorphism

$$\varphi_\psi : S \to T_\psi^{\mathrm{KR}}$$

  such that



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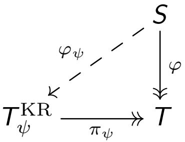

KR-cover

A profinite semigroup S is a KR-cover if it is a KR-cover of each of its finite continuous homomorphic images.

Examples

- **Groups**
  more generally: completely simple semigroups
- V-**projective profinite semigroups**
  for V closed under two-sided Karnofsky–Rhodes expansion
- **Free pro-**V **semigroups**
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Coproduct of KR-covers

Theorem (First closure theorem)

For every pseudovariety of semigroups V *closed under two-sided Karnofsky–Rhodes expansion, the class of all pro-*V *KR-covers is closed under* V*-coproducts*

Corollary

Every profinite coproduct of KR-covers is a KR-cover, whence equidivisible.

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Corollary

Every profinite coproduct of KR-covers is a KR-cover, whence equidivisible.

Adding the cancellation property

Theorem

Let S be a finitely generated profinite semigroup that is letter super-cancellative.
Then:

$$S \text{ is equidivisible} \Leftrightarrow S \text{ is a KR-cover}$$

Theorem (Second closure theorem)

For every pseudovariety of semigroups V closed under two-sided Karnofsky–Rhodes expansion, the class of letter super-cancellative equidivisible finitely generated pro-V semigroups is closed under finite V-coproducts.

Adding the cancellation property

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Let S be a finitely generated profinite semigroup that is letter super-cancellative.
Then:

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One example and a motivation

Example

The profinite semigroup $\varprojlim_{n\geq 1}\{0,1\}^{KR^n}$ is a KR-cover wich is letter super-cancellative, but not relatively free.

Theorem (Almeida et al., 2019)

Let S be an equidivisible profinite semigroup which is letter super-cancellative. Let $u, g \in S$. If $ug = u$ and g is regular (i.e., $g \in gSg$), then g is idempotent.

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Problems

- When does V-coproduct preserve equidivisibility of pro-V semigroups?
- Characterize the equidivisible (pro)finite semigroups that are KR-covers.
- Characterize the equidivisible profinite semigroups.