Algorithmic correspondence and analytic rules for (D)LE logics

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Analytic inductive ↔ analytic rules ↔ geometric formulas

$$\begin{tikzcd}
 & & & & & {[\,\mathrm{LR}\,]_{\dashv\vdash}} \arrow[d, leftrightarrow] & & & & \\
\text{modal formulas} & \supseteq & \text{analytic inductive} \arrow[rrrrr, leftrightarrow, bend left=60] & \ni & \varphi \arrow[ur, leftrightarrow, bend left, "\text{MASSA}"] \arrow[r, leftrightarrow, "\text{FO correspondent}", "\text{modal companion}"'] \arrow[dr, leftrightarrow, bend right, "\text{ALBA}"'] & \psi \arrow[d, leftrightarrow] & \in & \text{geom. formula} & \subseteq & \text{first-order formulas} \\
 & & & & & {[\,\mathrm{DR}\,]_{\dashv\vdash}} & & & &
\end{tikzcd}$$

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- A geometric theory is a FO theory whose models are preserved and reflected by geometric morphisms.

Analytic-inductive formulas (I)

Definition (**Signed Generation Tree**)

The **signed generation tree** of φ is defined by labelling the root of the syntax tree of φ with $+$ or $-$, and then propagating the labelling as follows:

- $\vee$, $\wedge$, $\Diamond$ or $\Box$: assign the same sign to its children.
- $\neg$: assign the opposite sign to its child (treat $s \rightarrow t$ as $\neg s \vee t$).

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Definition (**Order type**)

An **order type** is a map $\epsilon : \{p_1, \ldots, p_n\} \to \{1, \partial\}$.
An ϵ**-critical node** is a leaf node $+p_i$ with $\epsilon(p_i) = 1$ or $-p_i$ with $\epsilon(p_i) = \partial$.

Analytic-inductive formulas (II)

Definition (Definite analytic-inductive formula)

For any order type ϵ, and any strict linear order $<_\Omega$ on the variables, a formula is analytic (Ω, ϵ)-inductive if:

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- every branch is a concatenation of two paths P_1 and P_2 from leaf to root, such that P_1 consists of **PIA** nodes, i.e. $\{-\wedge, +\vee, -\Diamond, +\Box, + \rightarrow, \pm\neg\}$; and P_2 consists of **skeleton** nodes, i.e. $\{-\vee, +\wedge, +\Diamond, -\Box, - \rightarrow, \pm\neg\}$.

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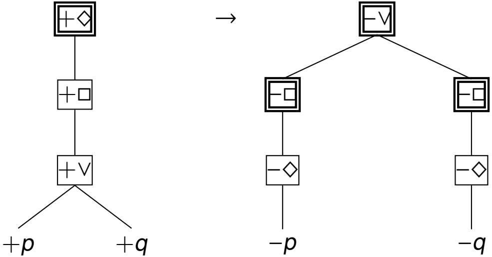
- each subtree rooted in a binary PIA node contains at most one ϵ-critical variable p and all the other variables q in the subtree are such that $q <_\Omega p$.

A formula $\varphi \rightarrow \psi$ is an **analytic-inductive** if $+\varphi$, $-\psi$ are analytic (Ω, ϵ)-inductive for some ϵ, $<_\Omega$.

Example: analytic inductive formula

$\Diamond\Box(p \vee q) \rightarrow \Box\Diamond p \vee \Box\Diamond q$, with $\epsilon(p) = \epsilon(q) = \partial$ and $p <_\Omega q$.



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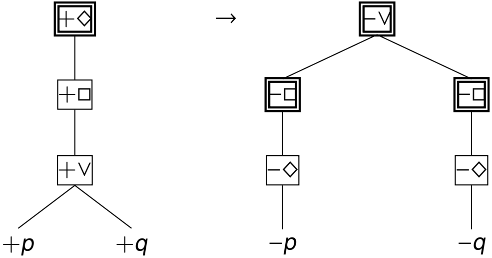

- **Not** analytic-inductive:

$$A \rightarrow \Box\Box\Diamond\Box\Diamond A$$

$$[\,\Box(\Diamond A \rightarrow \Box B) \wedge (\Box C \rightarrow A)\,] \rightarrow [\,\Diamond\Box C \vee A \vee (C \rightarrow B)\,]$$

The labelled Gentzen calculus G3K

Propositional rules

$$\wedge_L\ \frac{\Gamma, x{:}A, x{:}B \vdash \Delta}{\Gamma, x{:}A \wedge B \vdash \Delta} \qquad \frac{\Gamma \vdash x{:}A, \Delta \qquad \Gamma \vdash x{:}B, \Delta}{\Gamma \vdash x{:}A \wedge B, \Delta}\ \wedge_R$$

$$\vee_L\ \frac{\Gamma, x{:}A \vdash \Delta \qquad \Gamma, x{:}B \vdash \Delta}{\Gamma, x{:}A \vee B \vdash \Delta} \qquad \frac{\Gamma \vdash x{:}A, x{:}B, \Delta}{\Gamma \vdash x{:}A \vee B, \Delta}\ \vee_R$$

$$\rightarrow_L\ \frac{\Gamma \vdash x{:}A, \Delta \qquad \Gamma, x{:}B \vdash \Delta}{\Gamma, x{:}A \rightarrow B \vdash \Delta} \qquad \frac{\Gamma, x{:}A \vdash x{:}B, \Delta}{\Gamma \vdash x{:}A \rightarrow B, \Delta}\ \rightarrow_R$$

Modal rules (side condition on y in $\Box_R$ and $\Diamond_L$)

$$\Box_L\ \frac{xRy;\ \Gamma, x{:}\Box A, y{:}A \vdash \Delta}{xRy;\ \Gamma, x{:}\Box A \vdash \Delta} \qquad \frac{xRy;\ \Gamma \vdash y{:}A, \Delta}{\Gamma \vdash x{:}\Box A, \Delta}\ \Box_R$$

$$\Diamond_L\ \frac{xRy;\ \Gamma, y{:}A \vdash \Delta}{\Gamma, x{:}\Diamond A \vdash \Delta} \qquad \frac{xRy;\ \Gamma \vdash y{:}A, x{:}\Diamond A, \Delta}{xRy;\ \Gamma \vdash x{:}\Diamond A, \Delta}\ \Diamond_R$$

Example: derivation in G3K

$$
\dfrac{\dfrac{\dfrac{\Box_L\,\dfrac{\Box_L\,\dfrac{\to_L\,\dfrac{\mathrm{Id}_{y:B}\,\dfrac{}{xRy;\ y{:}B \vdash y{:}B} \qquad \dfrac{}{xRy;\ y{:}A \vdash y{:}A}\,\mathrm{Id}_{y:A}}{xRy;\ y{:}A \to B,\ y{:}A \vdash y{:}B}}{xRy;\ y{:}A \to B,\ x{:}\Box A \vdash y{:}B}}{xRy;\ x{:}\Box(A \to B),\ x{:}\Box A \vdash y{:}B}}{x{:}\Box(A \to B),\ x{:}\Box A \vdash x{:}\Box B}\,\Box_R}{x{:}\Box(A \to B) \vdash x{:}\Box A \to \Box B}\,\to_R}{\vdash x{:}\Box(A \to B) \to (\Box A \to \Box B)}\,\to_R
$$

The problem

- Given a modal formula φ, we want to find a rule R such that G3K $+$ R captures the logic $K + \varphi$.

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- Given a modal formula φ, we want to find a rule R such that G3K $+$ R captures the logic $K + \varphi$.
- R should be an **analytic** rule, that is, we should be able to add it to the calculus in a modular way (preserving cut elimination).
- The shape of a geometric formula is shown below. They can always be transformed into analytic rules.

$$\forall \overline{x}(P_1 \mathbin{\&} ... \mathbin{\&} P_m \rightarrow \exists y_{1_1}...y_{1_k} M_1 \vee ... \vee \exists y_{n_1}...y_{n_k} M_n)$$

The idea

- Let φ be not derivable in G3K. What is the **minimal** set of assumptions Γ that makes $\Gamma \vdash \varphi$ derivable? The obvious choice is $\Gamma = \varphi$.

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- Let φ be not derivable in G3K. What is the **minimal** set of assumptions Γ that makes $\Gamma \vdash \varphi$ derivable? The obvious choice is $\Gamma = \varphi$.
- The plan: we derive $x : \varphi \vdash x : \varphi$ and eliminate the red assumptions cutting on the atoms in the proof tree, preserving the relational information.

The algorithm (I)

Step I. Given the modal formula φ, construct a cut-free π_{φ} proof of the sequent $x : \varphi \vdash x : \varphi$ and propagate the colours to formulas following the rules.

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$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\mathrm{Id}_{t:A}$}
\UnaryInfC{$[1]\ xRy, yRt;\ t{:}{\color{blue}A} \vdash t{:}{\color{red}A}$}
\LeftLabel{$\Box_L$}
\UnaryInfC{$xRy, yRt;\ y{:}\Box{\color{blue}A} \vdash t{:}{\color{red}A}$}
\RightLabel{$\Box_R$}
\UnaryInfC{$xRy;\ y{:}\Box{\color{blue}A} \vdash y{:}\Box{\color{red}A}$}
\RightLabel{$\Diamond_R$}
\UnaryInfC{$xRy;\ y{:}\Box{\color{blue}A} \vdash x{:}\Diamond\Box{\color{red}A}$}
\LeftLabel{$\Diamond_L$}
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\AxiomC{}
\RightLabel{$\mathrm{Id}_{w:A}$}
\UnaryInfC{$[2]\ xRz, zRw;\ w{:}{\color{red}A} \vdash w{:}{\color{blue}A}$}
\RightLabel{$\Diamond_R$}
\UnaryInfC{$xRz, zRw;\ w{:}{\color{red}A} \vdash z{:}\Diamond{\color{blue}A}$}
\LeftLabel{$\Diamond_L$}
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\LeftLabel{$\Box_L$}
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\RightLabel{$\Box_R$}
\UnaryInfC{$x{:}\Box\Diamond{\color{red}A} \vdash x{:}\Box\Diamond{\color{blue}A}$}
\LeftLabel{$\to_L$}
\BinaryInfC{$x{:}\Diamond\Box{\color{blue}A}, x{:}\Diamond\Box{\color{red}A} \to \Box\Diamond{\color{red}A} \vdash x{:}\Box\Diamond{\color{blue}A}$}
\RightLabel{$\to_R$}
\UnaryInfC{$x{:}\Diamond\Box{\color{red}A} \to \Box\Diamond{\color{red}A} \vdash x{:}\Diamond\Box{\color{blue}A} \to \Box\Diamond{\color{blue}A}$}
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\RightLabel{$\Box_R$}
\UnaryInfC{$xRy;\ y{:}\Box A \vdash y{:}\Box A$}
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\UnaryInfC{$xRy;\ y{:}\Box A \vdash x{:}\Diamond\Box A$}
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\end{prooftree}
$$

Now all the information is stored in the leaves.
Note that we can always do this in an obvious way.

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Step II. Consider the leaves of π_φ and perform all possible cuts on atomic red formulas. Collect all the conclusions and use them as leaves in a forward-chaining proof search with goal $\vdash x : \varphi$. Collect all the attempts π_φ^i.

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$$\Box_L\ \cfrac{\cfrac{xRy, yRt, xRz, zRw, t{=}w;\ t{:}A \vdash w{:}A}{xRy, yRt, xRz, zRw, t{=}w;\ t{:}A \vdash z{:}\Diamond A}\ \Diamond_R}{xRy, yRt, xRz, zRw, t{=}w;\ y{:}\Box A \vdash z{:}\Diamond A}$$

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The process always terminates because $<_\Omega$ is well founded.

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Due to the nature of the G3K rules, this step can always be done, also you can not proceed further.

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\UnaryInfC{$xRy, yRt, xRz, zRw, t{=}w;\ y{:}\Box A \vdash z{:}\Diamond A$}
\dashedLine
\LeftLabel{Con}
\UnaryInfC{$xRz, xRy\,;\ y{:}\Box A \vdash z{:}\Diamond A$}
\LeftLabel{$\Diamond_L$}
\UnaryInfC{$xRz\,;\ x{:}\Diamond\Box A \vdash z{:}\Diamond A$}
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The definition of analytic-inductive formulas guarantees that the “merging point” is well-defined.

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$$Con \equiv \forall xyz\,(xRy \,\&\, xRz \Rightarrow \exists wt\,(yRt \,\&\, zRw \,\&\, t = w)) \equiv$$

$$\equiv \forall xyz\,(xRy \,\&\, xRz \Rightarrow \exists w\,(yRw \,\&\, zRw))$$

Conclusion and future work (I)

- We introduced an **algorithm** that associates **analytic-inductive formulas** with both their corresponding **analytic rules** and their **first-order correspondents**, using only the machinery of the G3K calculus...

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Can we do better?

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- **...** We can extend the approach from G3K to **every LE-logic** (nondistributive-lattice with finite-arity normal operators)**...**

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$$\frac{\Gamma, \overline{\mathbf{h} \leq A}, \overline{B \leq \mathbf{n}} \vdash \mathbf{j} \leq g(\overline{\mathbf{n}}, \overline{\mathbf{h}}), \Delta}{\Gamma \vdash \mathbf{j} \leq g(\overline{B}, \overline{A}), \Delta}\, g_R \qquad \frac{\Gamma, \overline{\mathbf{h} \leq A}, \overline{B \leq \mathbf{n}} \vdash f(\overline{\mathbf{h}}, \overline{\mathbf{n}}) \leq \mathbf{m}, \Delta}{\Gamma \vdash f(\overline{A}, \overline{B}) \leq \mathbf{m}, \Delta}\, f_L$$

$$\frac{\big(\Gamma \vdash \mathbf{h}_j \leq A_j, \Delta\big)_j \qquad \big(\Gamma \vdash B_i \leq \mathbf{n}_i, \Delta\big)_i}{\Gamma, \mathbf{j} \leq g(\overline{B}, \overline{A}) \vdash \mathbf{j} \leq g(\overline{\mathbf{n}}, \overline{\mathbf{h}}), \Delta}\, g_L \qquad \frac{\big(\Gamma \vdash \mathbf{h}_i \leq A_i, \Delta\big)_i \qquad \big(\Gamma \vdash B_j \leq \mathbf{n}_j, \Delta\big)_j}{\Gamma, f(\overline{A}, \overline{B}) \leq \mathbf{m} \vdash f(\overline{\mathbf{h}}, \overline{\mathbf{n}}) \leq \mathbf{m}, \Delta}\, f_R$$

(of course I need the appropriate labelled calculus)

Conclusion and future work (III)

- ... We will also consider a larger class of formulas, **inductive formulas**, that correspond to **systems of rules** and **generalized geometric formulas**.

$$GA_0 := \forall \overline{x}(\bigwedge P_i \to \exists y_1 \bigwedge M_1 \vee ... \vee \exists y_m \bigwedge M_m)$$

$$GA_{n+1} := \forall \overline{x}(\bigwedge P_i \to \exists y_1 \bigwedge GA_{k_1} \vee ... \vee \exists y_m \bigwedge GA_{k_m})$$

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- For example:

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Thanks!