Comparison of tabular intermediate logics

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Conventions

Convention (1)

By a poset we understand a finite nonempty partially ordered set.

Convention (2)

We consider posets as intuitionistic Kripke frames.

Convention (3)

By a formula we understand a propositional formula.

Intermediate logics

- Tabular intermediate logics posses semantics given by a finite frame P (here just a poset).
- There are intermediate logics without such property. For example the Gödel–Dummett logic.

p-morphism

Let $P = \langle W_P, \leq_P \rangle$ and $Q = \langle W_Q, \leq_Q \rangle$ be posets.
A map $h : W_P \to W_Q$ satisfying the following conditions:

(C1) h **preserves order**, i.e. if $a \leq_P b$, then $h(a) \leq_Q h(b)$,

(C2) h has **backward property**, i.e. if $\bar{a} \leq_Q \bar{b}$ and $h(a) = \bar{a}$, then there is $b \in W_P$ such that $a \leq_P b$ and $h(b) = \bar{b}$,

is called a *p-morphism* of P into Q.
If a p-morphism $h \colon W_P \to W_Q$ is surjective then Q is called a **p-morphic image** of P.

Truth preserving operations - p-morphic image

Let S_1, S_2 be posets. If S_2 is a p-morphic image of S_1 then for any formula ϕ:

$$S_1 \models \phi \implies S_2 \models \phi.$$

Truth preserving operations - generated subposet

Let S_1 be a poset. If S_2 is a generated subposet (up-set) of S_1 then for any formula ϕ:

$$S_1 \models \phi \implies S_2 \models \phi.$$

Truth preserving operations - disjoint union

Let $\{S_i\}_{i \in I}$ be the nonempty family of posets. For any formula ϕ holds:

$$\biguplus_{i \in I} S_i \models \phi \iff \forall i \in I : \; S_i \models \phi.$$

Relation $\preceq$

- Let S and T be posets. We write $S \preceq T$ if and only if S is a p-morphic image of some generated subposet of T.

Jankov - de Jongh Theorem

For every rooted (and finite) poset S there is a formula $\chi(S)$ such that for any poset T it holds that:

$$T \not\models \chi(S) \iff S \preceq T.$$

We call the formula $\chi(S)$ **de Jongh's formula** for the poset S.

Int-Log-Contain and Int-Log-Equal

- **INSTANCE:** Two finite frames (posets): P and Q.
- **QUESTION:** Does $L(P) \subseteq L(Q)$?
  - **Int-Log-Equal:** Does $L(P) = L(Q)$?

Int-Log-Contain is in NP class

Theorem

Let P and Q be finite frames. Then $L(P) \subseteq L(Q)$ iff every rooted generated subframe of Q is a p-morphic image of a rooted generated subframe of P.

This shows that **Int-Log-Contain** is in the **NP** complexity class. It also shows that the following problem is trivially reducible to **Int-Log-Contain**.

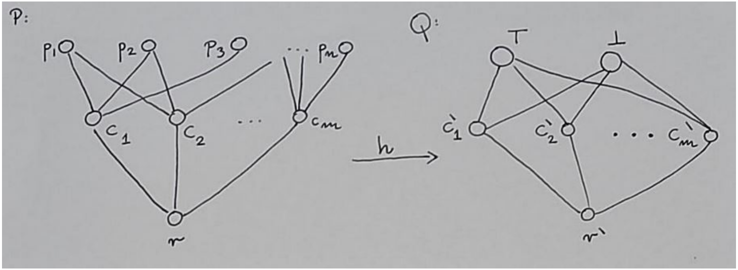
Monotone NAE-3-SAT

- The abbreviation **Monotone NAE-3-SAT** stands for **Monotone Not All Equal - 3 - Satisfiability Problem**.
- The instance of the problem consists of the finite set of boolean variables $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ and the finite family of clauses $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$.
- Each clause is built of three different variables from $\mathcal{P}$.
- In the problem we require such a valuation on $\mathcal{P}$ so each clause consists of at least one true and at least one false value.

p-Image-Gen-Sub Problem

- **INSTANCE:** Two finite rooted frames P and Q,
- **QUESTION:** Does there exist a surjective p-morphism from a generated subframe of P onto Q ?

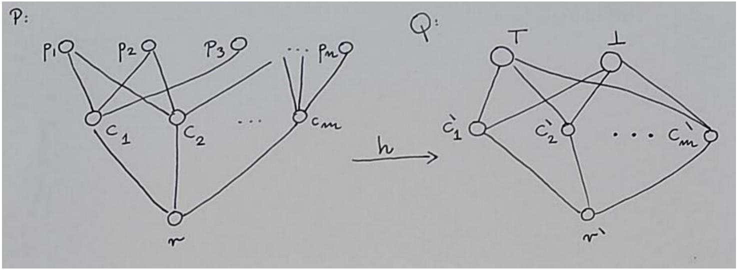
p-Image-Gen-Sub is in NP-hard (1)



Remarks:

- $r \leq_P c_i$ *for all* i,
- $r' \leq_Q c'_i$ *for all* i,
- each c_i is below exactly 3 different p_j,
- $c'_i \leq_Q \bot$ and $c'_i \leq_Q \top$ (*for all* i).

p-Image-Gen-Sub is in NP-hard (2)



Remarks:

- $h(c_i) = c'_i$,
- $h(r) = r'$,

and hence:

- $h(\{p_1, \ldots, p_m\}) \subseteq \{\bot, \top\}$.

p-Image

- **INSTANCE:** Two finite rooted frames: P and Q.
- **QUESTION:** Does there exist a surjective p-morphism from P onto Q ?

Conclusion

- **p-Image** is a generalization of **p-Image-Gen-Sub**.
- Hence, **p-Image** is in NP-complete class.

Procedure of comparison

The Procedure finds whether logic of one poset is contained in logic of the other. Input arguments are two posets, P and Q. The procedure returns one of the values **YES** or **NO** where **YES** means that $L(P) \subseteq L(Q)$, while **NO** denotes that $L(P) \not\subseteq L(Q)$:

1. $\mathcal{A} \leftarrow M(Q)$, $\mathcal{B} \leftarrow M(P)$.
2. If $\mathcal{A} = \emptyset$ then return **YES**.
3. Take any $U \in \mathcal{A}$. If there exists $U' \in \mathcal{B}$ such that $U \preceq U'$ then $\mathcal{A} \leftarrow \mathcal{A} \setminus \{U\}$ and go to the step **2.** If such U' does not exist the return **NO**.

Remark

- There is correspondence to **CSP** problem.

Further research

- Applying a constraint on the width (maximal antichain) of the target poset may result in polynomial-algorithm solution.

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THANK YOU FOR YOUR ATTENTION !