Torsion theories and coverings of preordered groups

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Overview

Preordered group

Definition

A **preordered group** $(G, \leq)$ is a group $(G, +, 0)$ endowed with a preorder relation $\leq$ on G which is compatible with $+$:

$$a \leq c \text{ and } b \leq d \;\Rightarrow\; a + b \leq c + d \qquad \text{for } a, b, c, d \in G.$$

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Example

The group $\mathbb{Z}$ of integers with the usual order $\leq$: $(\mathbb{Z}, \leq)$
$(\mathbb{Z}, \leq)$ is a ***partially ordered group****.*

Morphism of preordered groups

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A **morphism of preordered groups** $f\colon (G, \leq_G) \to (H, \leq_H)$ is a group morphism $f\colon G \to H$ which preserves the preorder:

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All preordered groups and morphisms between them define a category denoted by **PreOrdGrp**.

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  Notation: (G, P_G)
- ***arrows***: *pairs* $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ *making the following square commute:*

$$\begin{array}{ccc} P_G & \xrightarrow{\bar{f}} & P_H \\ \downarrow & & \downarrow \\ G & \xrightarrow[f]{} & H \end{array}$$

Some interesting full subcategories

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  Equivalently: $G = G$.

Kernels, cokernels and short exact sequences

Proposition [M.M. Clementino, N. Martins-Ferreira, A. Montoli (2019)]

Consider, in PreOrdGrp, *a pair of composable arrows as in the following diagram:*

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\begin{array}{ccccc}
P_A & \xrightarrow{\bar{k}} & P_B & \xrightarrow{\bar{f}} & P_C \\
\downarrow & (P) & \downarrow & & \downarrow \\
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Then:

- $(k, \bar{k}) = \ker(f, \bar{f})$ *if and only if* $k = \ker(f)$ *in* Grp *and* (P) *is a pullback in* Mon.

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Proposition (second part)

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- $(f, \bar{f}) = \mathrm{coker}(k, \bar{k})$ *if and only if* $f = \mathrm{coker}(k)$ *in* Grp *and* $\bar{f}$ *is surjective;*

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- $(f, \bar{f}) = \mathrm{coker}(k, \bar{k})$ *if and only if* $f = \mathrm{coker}(k)$ *in* Grp *and* $\bar{f}$ *is surjective;*
- (1) *is a short exact sequence in* PreOrdGrp *if and only if* $A \xrightarrow{k} B \xrightarrow{f} C$ *is a short exact sequence in* Grp, (P) *is a pullback in* Mon *and* $\bar{f}$ *is surjective.*

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Reminder

A category $\mathscr{C}$ *is* ***normal*** *[Z. Janelidze (2010)] when*

- *it has a zero object* 0*;*
- *it is regular;*
- *any regular epimorphism is a normal epimorphism.*

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Proposition

The pair of full (replete) subcategories (Grp, ParOrdGrp) *of* PreOrdGrp *is a torsion theory in the normal category* PreOrdGrp*.*

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1. *the only arrow from any* $T \in \mathscr{T}$ *to any* $F \in \mathscr{F}$ *is the zero arrow;*
2. *for any object* $C \in \mathscr{C}$ *there exists a short exact sequence*

$$0 \longrightarrow T \xrightarrow{\epsilon_C} C \xrightarrow{\eta_C} F \longrightarrow 0$$

with $T \in \mathscr{T}$ *and* $F \in \mathscr{F}$.

A torsion theory in PreOrdGrp: proof

Sketch of the proof

Let $(G, P_G) \in \mathsf{PreOrdGrp}$ *and define*

$$N_G = \{n \in G \mid n \in P_G \textit{ and } -n \in P_G\}.$$

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N_G *is a normal subgroup of* G *so that the sequence*

$$N_G \overset{k_G}{\rightarrowtail} G \overset{\eta_G}{\twoheadrightarrow} G/N_G$$

is a short exact sequence in Grp.

A torsion theory in PreOrdGrp: proof

Sketch of the proof

$$
\begin{array}{ccccc}
N_G & \overset{\bar{k}_G}{\rightarrowtail} & P_G & \overset{\bar{\eta}_G}{\twoheadrightarrow} & \eta_G(P_G) \\
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By construction this sequence is a short exact sequence in PreOrdGrp*.*

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By construction this sequence is a short exact sequence in PreOrdGrp.

It remains to prove:

- $(N_G, N_G) \in$ Grp;
- $(G/N_G, \eta_G(P_G)) \in$ ParOrdGrp.

Consequence of the torsion theory

Reminder

Any torsion theory $(\mathscr{T}, \mathscr{F})$ in a normal category $\mathscr{C}$ induces two functors:

- $F : \mathscr{C} \to \mathscr{F}$ *is a* ***(normal epi)-reflector****;*
- $T : \mathscr{C} \to \mathscr{T}$ *is a* ***(normal mono)-coreflector****.*

Consequence of the torsion theory

Corollary

The category ParOrdGrp *is reflective in* PreOrdGrp

$$\mathsf{PreOrdGrp} \underset{U}{\overset{F}{\underset{\hookleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{ParOrdGrp}$$

and each component of the unit η of the adjunction is a normal epimorphism.

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The category ParOrdGrp *is reflective in* PreOrdGrp

$$\mathrm{PreOrdGrp} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{ParOrdGrp} \quad (F \dashv U)$$

and each component of the unit η *of the adjunction is a normal epimorphism.*

If $\mathscr{X}$ denotes the class of **all morphisms** in PreOrdGrp, then

$$\Gamma = (\mathrm{PreOrdGrp}, \mathrm{ParOrdGrp}, F, U, \mathscr{X})$$

forms a **Galois structure**.

Semi-left-exact reflector

Reminder [C. Cassidy, M. Hébert, G.M. Kelly (1985)]

*A reflector $F\colon \mathscr{C} \to \mathscr{F}$ is said to be **semi-left-exact** when it preserves all pullbacks of the form*

$$\begin{array}{ccc} P & \longrightarrow & U(C) \\ \downarrow & & \downarrow U(f) \\ B & \xrightarrow[\eta_B]{} & UF(B) \end{array}$$

where $\eta_B\colon B \to UF(B)$ is the B-component of the unit of the reflection $F \dashv U$ and $f\colon C \to F(B)$ is in the subcategory $\mathscr{F}$.

Semi-left-exact reflector

Proposition

The reflector $F\colon \mathsf{PreOrdGrp} \to \mathsf{ParOrdGrp}$ *in the adjunction*

$$\mathsf{PreOrdGrp} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{ParOrdGrp} \quad (F \dashv U)$$

is semi-left-exact.

Link with the admissibility

Reminder [G. Janelidze (1990)]

Let $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathscr{X})$ be a Galois structure (where $F \dashv U$ is a full reflection).
*Then $F \colon \mathscr{C} \to \mathscr{F}$ is **semi-left-exact** if and only if Γ is **admissible** in the sense of categorical Galois theory.*

Induced factorization system

Reminder [C. Cassidy, M. Hébert, G.M. Kelly (1985)]

If a category $\mathscr{C}$ has a full reflective subcategory $\mathscr{F}$

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*such that the reflector F is **semi-left-exact**, we then naturally get a factorization system $(\mathscr{E}, \mathscr{M})$ defined as follows:*

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In this context $\mathscr{M}$ is the class of **trivial coverings**.

Consequences

Corollary

- *The Galois structure* $\Gamma = (\mathsf{PreOrdGrp}, \mathsf{ParOrdGrp}, F, U, \mathscr{X})$ *is admissible.*
- *We naturally get a factorization system* $(\mathscr{E}, \mathscr{M})$*.*

The classes $\mathscr{E}'$ and $\mathscr{M}^*$

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Reminder [A. Carboni, G. Janelidze, G.M. Kelly, R. Paré (1997)]

A factorization system is said to be ***monotone-light*** *when it is of the form $(\mathscr{E}', \mathscr{M}^*)$ for some factorization system $(\mathscr{E}, \mathscr{M})$.*

Characterization of $\mathscr{E}'$ and $\mathscr{M}^*$ in PreOrdGrp

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In PreOrdGrp, *the pair* $(\mathscr{E}', \mathscr{M}^*)$ *is a monotone-light factorization system, and*

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Characterization of $\mathscr{E}'$ and $\mathscr{M}^*$ in PreOrdGrp: proof

Proof

Proof of 2 Propositions (Condition (N) + Condition (C))
+ application of 1 Theorem [T. Everaert, M. Gran (2013)]

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Let $(\mathscr{T}, \mathscr{F})$ *be a torsion theory in a normal category* $\mathscr{C}$ *and* $(\mathscr{E}, \mathscr{M})$ *the factorization system associated with the (semi-left-exact) reflector* $F \colon \mathscr{C} \to \mathscr{F}$.

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Assume that the following two conditions hold:

(N) *for any normal monomorphism* $k \colon K \to A$*, the monomorphism* $k \cdot \epsilon_K \colon T(K) \to A$ *is normal, where* $\epsilon_K \colon T(K) \to K$ *is the* K*-component of the counit* ϵ *of the coreflection* $T \colon \mathscr{C} \to \mathscr{T}$*;*

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Then $(\mathscr{E}', \mathscr{M}^*)$ *is a monotone-light factorization system, and*

- $\mathscr{E}' = \{f \in \mathscr{C} \mid f$ *is a normal epimorphism, and* $\mathrm{Ker}(f) \in \mathscr{T}\}$*;*
- $\mathscr{M}^* = \{f \in \mathscr{C} \mid \mathrm{Ker}(f) \in \mathscr{F}\}$*.*

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Corollary

The coverings with respect to the adjunction

$$\mathsf{PreOrdGrp} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{ParOrdGrp} \quad (F \dashv U)$$

are the morphisms $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ *in* PreOrdGrp *such that* $\mathsf{Ker}(f, \bar{f}) \in \mathsf{ParOrdGrp}$.

- Besides the torsion theory mentioned above there is also in PreOrdGrp a **pretorsion theory** (in the sense of A. Facchini and C. Finocchiaro) given by the pair

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$$(\mathsf{Grp}(\mathsf{PreOrd}), \mathsf{ParOrdGrp}).$$

- The coverings described above can be **classified** in terms of internal actions of a Galois groupoid.

- Besides the torsion theory mentioned above there is also in PreOrdGrp a **pretorsion theory** (in the sense of A. Facchini and C. Finocchiaro) given by the pair

$$(\mathsf{Grp}(\mathsf{PreOrd}), \mathsf{ParOrdGrp}).$$

- The coverings described above can be **classified** in terms of internal actions of a Galois groupoid.

- The results presented in the setting of preordered groups can be generalized to **V-groups** for V a suitable quantale (e.g. Lawvere metric groups, Lawvere ultrametric groups, probabilistic metric groups, etc.).

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