Difference–restriction algebras of partial functions with operators: discrete duality

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(with Célia Borlido)

Recap (of Célia's talk)

$-$ is **relative complement**: $f - g := \{(x, y) \mid (x, y) \in f \text{ and } (x, y) \notin g\}$.
$\triangleright$ is **domain restriction**: $f \triangleright g := \{(x, y) \mid x \in \mathrm{dom}(f) \text{ and } (x, y) \in g\}$.

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The representable algebras form a finitely based variety and the *completely representable* algebras are the subclass of *atomic* representable algebras.

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- Discrete duality
- Extension with additional operators

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  The category **AtRepAlg**:

  - *objects*: atomic and representable algebras of the signature $\{-, \triangleright\}$,
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- Compatibly complete, compatible completion, duality

- Extension of results to algebras equipped with additional operators

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1. φ **preserves equivalence**: if both $\varphi(x)$ and $\varphi(x')$ are defined, then

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In particular, φ induces a partial function $\widetilde{\varphi}\colon X_0 \rightharpoonup Y_0$ given by

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2. φ is **fibrewise injective**: for every $(x_0, y_0) \in \widetilde{\varphi}$, the restriction and co-restriction of φ induces an injective partial map
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3. φ is **fibrewise surjective**: for every $(x_0, y_0) \in \widetilde{\varphi}$, the induced partial map $\varphi_{(x_0,y_0)}$ is surjective (that is, the image of $\varphi_{(x_0,y_0)}$ is the whole of $\rho^{-1}(y_0)$).

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For $y \in \mathrm{At}(\mathfrak{B})$, *if* there is $a \in \mathfrak{A}$ with $h(a) \geq y$ then there is a unique atom x with $h(x) \geq y$. When this happens, define $Fh(y) = x$, otherwise undefined.

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Given a morphism φ from $(\pi\colon X \twoheadrightarrow X_0)$ to $(\rho\colon Y \twoheadrightarrow Y_0)$ in $\mathbf{Set_q}$

$$G\varphi(g) = \{(\pi(x), x) \in X_0 \times X \mid \exists y \in Y\colon (x, y) \in \varphi \text{ and } (\rho(y), y) \in g\}.$$

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...and notion of completeness (w.r.t. joins) should demand joins of *all* such sets.

Compatibility: a general notion

Definition

Let $\mathfrak{P}$ be a poset. A binary relation C on $\mathfrak{P}$ is a **compatibility relation** if it is reflexive, symmetric, and downward closed in $\mathfrak{P} \times \mathfrak{P}$. We say that two elements $a_1, a_2 \in \mathfrak{P}$ are **compatible** if $a_1 C a_2$.

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Proposition

Let $(P, \leq, C)$ be a poset equipped with a binary relation C. Then $(P, \leq, C)$ is isomorphic to a poset $(P', \subseteq, C')$ of partial functions ordered by inclusion and equipped with the relation 'agree on the intersection of their domains' if and only if C is reflexive, symmetric, and downward closed.

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Proof:

$$\theta(p) := \{(\{p'\}, p') \mid p' \leq p\} \cup \{(\{p', q\}, p') \mid p' \leq p \text{ and } p' \not\subset q\}.$$

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Similarly *compatibly complete* $\implies$ *directed complete*.

Compatibly complete (continued)

For representable $\{-, \triangleright\}$-algebras:

compatibly complete $\implies$ *meet complete*.

The converse is false.

The three-element $\{-, \triangleright\}$-algebra consisting of the partial functions $\emptyset$, $\{(1,1)\}$, and $\{(2,2)\}$ provides a counterexample.

Compatible completions

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A **compatible completion** of a representable $\{-, \triangleright\}$-algebra $\mathfrak{A}$ is an embedding $\iota\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ of $\{-, \triangleright\}$-algebras such that $\mathfrak{C}$ is representable and compatibly complete and $\iota[\mathfrak{A}]$ is join dense in $\mathfrak{C}$.

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Lemma

Let $\iota\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ be an embedding of representable $\{-, \triangleright\}$-algebras. If $\iota[\mathfrak{A}]$ is join dense in $\mathfrak{B}$ then ι is complete.

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(So we may say 'the' compatible completion.)

Compatible completion using the adjunction

Theorem

For every atomic representable $\{-, \triangleright\}$-algebra $\mathfrak{A}$, the homomorphism

$$\begin{aligned}\eta_{\mathfrak{A}} \colon \mathfrak{A} &\to (G \circ F)(\mathfrak{A}) = \{f \colon \mathrm{At}(\mathfrak{A})/{\sim_{\mathfrak{A}}} \rightharpoonup \mathrm{At}(\mathfrak{A}) \mid f \subseteq \pi_{\mathfrak{A}}^{-1}\} \\ a &\mapsto \{([x], x) \mid x \in \mathrm{At}(\mathfrak{A}) \text{ and } x \leq a\}\end{aligned}$$

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Corollary

There is a duality between **CAtRepAlg** *and* $\mathbf{Set_q}$, *where* **CAtRepAlg** *is the full subcategory of* **AtRepAlg** *consisting of the compatibly complete algebras.*

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CAtRepAlg *is a reflective subcategory of* **AtRepAlg**.

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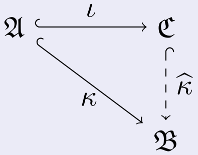
"compatible completions in **RepAlg**$_\infty$ are unique, and we know how to construct them for atomic algebras"

Completion in cat. with complete homomorphisms (cont.)

Proposition

Let $\iota\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ *be a complete embedding of representable* $\{-, \triangleright\}$*-algebras. Consider the following statements about* ι*.*

1. $\mathfrak{C}$ *is compatibly complete, and the image of* $\mathfrak{A}$ *is join dense in* $\mathfrak{C}$*.*
2. $\mathfrak{C}$ *is the 'smallest' extension of* $\mathfrak{A}$ *that is compatibly complete. That is,* $\mathfrak{C}$ *is compatibly complete, and for every other complete embedding* $\kappa\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ *into a compatibly complete and representable* $\{-, \triangleright\}$*-algebra* $\mathfrak{B}$*, there exists a complete embedding* $\widehat{\kappa}\colon \mathfrak{C} \hookrightarrow \mathfrak{B}$ *making the following diagram commute.*



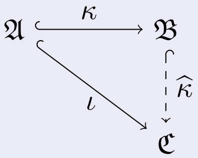
. . .

Completion in cat. with complete homomorphisms (cont.)

Proposition

. . .

3. *$\mathfrak{C}$ is the 'largest' extension of $\mathfrak{A}$ in which the image of $\mathfrak{A}$ is join dense. That is, $\iota[\mathfrak{A}]$ is join dense in $\mathfrak{C}$, and for every other complete embedding $\kappa\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ into a representable $\{-, \triangleright\}$-algebra $\mathfrak{B}$ in which the image of $\mathfrak{A}$ is join dense, there exists a complete embedding $\widehat{\kappa}\colon \mathfrak{B} \hookrightarrow \mathfrak{C}$ making the following diagram commute.*



Then 1 $\implies$ 2, and 1 $\implies$ 3, and if $\mathfrak{A}$ has a completion then all three conditions are equivalent.

Completion in cat. with complete homomorphisms (cont.)

Proposition

...

3. $\mathfrak{C}$ *is the 'largest' extension of* $\mathfrak{A}$ *in which the image of* $\mathfrak{A}$ *is join dense. That is,* $\iota[\mathfrak{A}]$ *is join dense in* $\mathfrak{C}$*, and for every other complete embedding* $\kappa\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ *into a representable* $\{-, \triangleright\}$*-algebra* $\mathfrak{B}$ *in which the image of* $\mathfrak{A}$ *is join dense, there exists a complete embedding* $\widehat{\kappa}\colon \mathfrak{B} \hookrightarrow \mathfrak{C}$ *making the following diagram commute.*

$$\begin{array}{ccc} \mathfrak{A} & \overset{\kappa}{\hookrightarrow} & \mathfrak{B} \\ & {\scriptstyle\iota}\searrow & \downarrow{\scriptstyle\widehat{\kappa}} \\ & & \mathfrak{C} \end{array}$$

Then $1 \implies 2$*, and* $1 \implies 3$*, and if* $\mathfrak{A}$ *has a completion then all three conditions are equivalent.*

(The **RepAlg** version of this statement does not hold.)

Additional operators

Definition

Let Ω be an n-ary operation on $\mathfrak{A}$. Then Ω is **compatibility preserving** if: a_i, a'_i compatible, for all $i \implies \Omega(a_1, \ldots, a_n)$, $\Omega(a'_1, \ldots, a'_n)$ compatible.
Ω is **completely additive** if whenever the supremum $\sum S$ exists, for $S \subseteq \mathfrak{A}$,

$$\Omega(a_1, \ldots, a_{i-1}, \sum S, a_{i+1}, \ldots, a_n) = \sum \Omega(a_1, \ldots, a_{i-1}, S, a_{i+1}, \ldots, a_n)$$

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Definition

The category **AtRepAlg**(σ) has

- *objects*: algebras of the signature $\{-, \triangleright\} \cup \sigma$ whose $\{-, \triangleright\}$-reduct is atomic and representable, and such that the symbols of σ are interpreted as compatibility preserving completely additive operations,
- *morphisms*: complete homomorphisms of $(\{-, \triangleright\} \cup \sigma)$-algebras.

Dually: additional relations

From compatibility preserving and completely additive n-ary Ω, can define $(n+1)$-ary relation R_Ω on atoms of $\mathfrak{A}$:

$$R_\Omega x_1 \ldots x_{n+1} \iff \Omega(x_1, \ldots, x_n) \geq x_{n+1}.$$

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Definition

Given: sets X, X_0, surjection $\pi : X \twoheadrightarrow X_0$, and R an $(n+1)$-ary relation on X.
The **compatibility relation** $C \subseteq X \times X$ is given by

$x \; C \; y$ if and only if $\pi(x) = \pi(y) \implies x = y$.

Then R has the **compatibility property** (with respect to π) if given $x_1 C x_1', \ldots, \; x_n C x_n'$ and $R x_1 \ldots x_{n+1}$ and $R x_1' \ldots x_{n+1}'$, we have $x_{n+1} C x_{n+1}'$.

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Given R satisfying compatibility property, can define n-ary operation Ω_R on the dual $\mathfrak{A}_\pi$ of $\pi : X \twoheadrightarrow X_0$ by conflating elements of $\mathfrak{A}_\pi$ with their image, and setting

$$\Omega_R(X_1, \ldots, X_n) = \bigcup_{x_1 \in X_1, \ldots, x_n \in X_n} \{x_{n+1} \in X \mid R x_1 \ldots x_{n+1}\}.$$

Morphisms and the dual category

Definition

Given: $\varphi \colon X \rightharpoonup Y$ and $(n+1)$-ary relations R_X and R_Y on X and Y.
Then φ satisfies the **reverse forth condition** if whenever $R_X x_1 \ldots x_{n+1}$ and $\varphi(x_1), \ldots, \varphi(x_n)$ are defined, then $\varphi(x_{n+1})$ is defined and $R_Y \varphi(x_1) \ldots \varphi(x_{n+1})$.
And φ satisfies the **back condition** if whenever $\varphi(x_{n+1})$ is defined and $R_Y y_1 \ldots y_n \varphi(x_{n+1})$, then there exist $x_1, \ldots, x_n \in \mathrm{dom}(\varphi)$ such that $\varphi(x_1) = y_1, \ldots, \varphi(x_n) = y_n$ and $R_X x_1 \ldots x_{n+1}$.

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Definition

The category $\mathbf{Set_q}(\sigma)$ has

- *objects*: the objects of $\mathbf{Set_q}$ equipped with, for each $\Omega \in \sigma$, an $(n+1)$-ary relation R_Ω that has the compatibility property, where n is the arity of Ω,
- *morphisms*: morphisms of $\mathbf{Set_q}$ that satisfy the reverse forth condition and the back condition with respect to R_Ω, for every $\Omega \in \sigma$.

Extended adjunction

Theorem

There is an adjunction $F' \colon \mathbf{AtRepAlg}(\sigma) \dashv \mathbf{Set_q}(\sigma)^{\mathsf{op}} : G'$ *extending* $F \colon \mathbf{AtRepAlg} \dashv \mathbf{Set_q}^{\mathsf{op}} : G$ *in the sense that the appropriate reducts of* $F'(\mathfrak{A})$ *and* $G'(\pi \colon X \twoheadrightarrow X_0)$ *equal* $F(\mathfrak{A})$ *and* $G(\pi \colon X \twoheadrightarrow X_0)$*, respectively.*

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Corollary

For every algebra $\mathfrak{A}$ *in* $\mathbf{AtRepAlg}(\sigma)$*, the embedding* $\eta_{\mathfrak{A}} \colon \mathfrak{A} \hookrightarrow (G' \circ F')(\mathfrak{A})$ *is the compatible completion of* $\mathfrak{A}$*.*

Corollary

There is a duality between $\mathbf{CAtRepAlg}(\sigma)$ *and* $\mathbf{Set_q}(\sigma)^{\mathrm{op}}$*, where* $\mathbf{CAtRepAlg}(\sigma)$ *is the full subcategory of* $\mathbf{AtRepAlg}(\sigma)$ *consisting of the compatibly complete algebras.*

Corollary

The category $\mathbf{CAtRepAlg}(\sigma)$ *is a reflective subcategory of* $\mathbf{AtRepAlg}(\sigma)$*.*

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- Which representable $\{-, \triangleright\}$-algebras have a compatible completion in $\mathbf{RepAlg}_{\infty}$? Describe a general method to construct these completions.

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- Which representable $\{-, \triangleright\}$-algebras have a compatible completion in $\mathbf{RepAlg}_\infty$? Describe a general method to construct these completions.
- Weaken the base signature $\{-, \triangleright\}$.
- Relax constraints on additional operators.
- Find a non-discrete duality for full class of representable algebras.

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