# Frobenius structure in ($*-$)autonomous categories

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# Motivations

**Theorem (Kruml and Paseka 2008, Santocanale 2020)**
Let $L$ be a complete lattice. The following are equivalent:

- $L$ is a completely distributive lattice.
- The set of join-preserving endomaps of $L$ is a Frobenius quantale.

**Theorem (Raney 1960, Higgs and Rowe 1989)**
The nuclear objects of the category of complete sup-lattices are exactly the completely distributive lattice.

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**Conjecture**

Let $L$ be an object of an autonomous category (symmetric monoidal closed). The following are equivalent:

- $L$ is nuclear.
- The object of endomorphisms of $L$ is a Frobenius structure.

**Theorem (Raney 1960, Higgs and Rowe 1989)**

The nuclear objects of the category of complete sup-lattices are exactly the completely distributive lattices.

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# Symmetric monoidal closed categories

## Definition

A symmetric monoidal category $(\mathcal{C}, \otimes, I, \rho, \lambda, \sigma)$ is *closed* (or *autonomous*) if there is a natural bijection:

$$\frac{X \otimes Y \longrightarrow Z}{Y \longrightarrow [X, Z]}$$

For an object 0 of $\mathcal{C}$, we write $(-)^* = [-, 0] : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$.
If the natural transformation $j_A : A \to A^{**}$ is an iso, then $\mathcal{C}$ is $*$-*autonomous*.

## Examples

- Autonomous categories: **Set**, $k$-**Vect**, a commutative unital quantale, *etc.*
- $*$-autonomous categories: $k$-$\mathbf{Vect}_{\mathrm{fin}}$, **SLatt**, *etc.*

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# Dual pair

For an object $A$ of a $*$-autonomous category, we have the two equivalences:

$$\frac{A \otimes X \longrightarrow 0}{X \longrightarrow A^*} \qquad \frac{X \otimes A^* \longrightarrow 0}{X \longrightarrow A^{**} \cong A\,.}$$

### Definition

A map $\epsilon : A \otimes B \longrightarrow 0$ in $\mathcal{V}$ is said to be a *dual pairing* (w.r.t. the object 0) if the two induced natural transformations are isomorphims.

$$\hom(X,B) \longrightarrow \hom(A \otimes X, 0)\,, \qquad \hom(X,A) \longrightarrow \hom(X \otimes B, 0)\,.$$

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# Some properties of dual pairs

**Proposition**

Let $(A, B)$ be a dual pair in a symmetric monoidal closed category.

1. $(B, A)$ is also a dual pair.
2. We have $A \cong B^*$.
3. $A$ is a reflexive object (*i.e* $A \cong A^{**}$).
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# Examples of dual pairs

**Examples**

- In **SLatt**, $(L, L^{\mathrm{op}}, \epsilon)$, $\epsilon(x, y) = \perp$ if $x \leq y$, and $\epsilon(x, y) = \top$ otherwise.
- In a $*$-autonomous category, $A^* \otimes A \cong [A, A]^*$ so $(A^* \otimes A, [A, A], \epsilon)$ is a dual pair with $\epsilon := ev \circ \sigma \circ ev$.

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# Usual adjunction between lattices

For a join preserving map $f : L \to M$, the right adjoint to it $\widetilde{f} : M^{\mathrm{op}} \to L^{\mathrm{op}}$ is the only map s.t:

$$f(x) \leq y \qquad \text{iff} \qquad x \leq \widetilde{f}(y)$$

$$\begin{tikzcd} L \otimes M^{\mathrm{op}} \arrow[r, "f \otimes M^{\mathrm{op}}"] \arrow[d, "L \otimes \widetilde{f}"'] & M \otimes M^{\mathrm{op}} \arrow[d, "\epsilon_M"] \\ L \otimes L^{\mathrm{op}} \arrow[r, "\epsilon_L"] & 0\,. \end{tikzcd}$$

## Adjoints in dual pair

Let $(A_0, B_0)$, $(A_1, B_1)$ be two dual pairs. For every morphism $f : A_0 \longrightarrow A_1$ we define $\widetilde{f} : B_1 \longrightarrow B_0$ by transposing:

$$\frac{\dfrac{A_0 \longrightarrow A_1}{A_0 \otimes B_1 \longrightarrow 0}}{B_1 \longrightarrow B_0}$$

$$\begin{tikzcd}
A_0 \otimes B_1 \arrow[r, "f \otimes B_1"] \arrow[d, "A_0 \otimes \widetilde{f}"'] & A_1 \otimes B_1 \arrow[d, "\epsilon_1"] \\
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# Next

# The category of semigroups over a monoidale category

Objects of $\mathbf{Sem}_{\mathcal{C}}$: pairs $(A, \mu_A)$ such that

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "A \otimes \mu_A"] \arrow[d, "\mu_A \otimes A"'] & A \otimes A \arrow[d, "\mu_A"] \\
A \otimes A \arrow[r, "\mu_A"'] & A.
\end{tikzcd}$$

Morphisms of $\mathbf{Sem}_{\mathcal{C}}$: arrows $f : A_0 \longrightarrow A_1$ such that

$$\begin{tikzcd}
A_0 \otimes A_1 \arrow[r, "f \otimes f"] \arrow[d, "\mu_{A_0}"'] & A_1 \otimes A_1 \arrow[d, "\mu_{A_1}"] \\
A_0 \arrow[r, "f"'] & A_1.
\end{tikzcd}$$

# Quantales

**Definition**

A *quantale* $(Q, \star)$ is a semigroup in the category **SLatt**.

**Remark**

In a quantale, $(x \star -) : Q \to Q$ and $(- \star y) : Q \to Q$ both have a right adjoint, the left and right implications:

$$x \star y \leq z \quad \text{iff} \quad y \leq x \backslash z \quad \text{iff} \quad x \leq z / y$$

We have

$$-/- : Q \otimes Q^{\mathrm{op}} \longrightarrow Q^{\mathrm{op}} \qquad \text{and} \qquad -\backslash - : Q^{\mathrm{op}} \otimes Q \longrightarrow Q^{\mathrm{op}}$$

$$z/(y \star x) = (z/y)/x \qquad \text{and} \qquad (x \star y)\backslash z = x \backslash (y \backslash z)$$

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# Implications in a quantale

$x \star y \leq z$ iff $x \leq z/y$

$$\begin{tikzcd}
Q \otimes Q \otimes Q^{\mathrm{op}} \arrow[r, "Q\otimes -/-"] \arrow[d, "\star\otimes Q^{\mathrm{op}}"'] & Q \otimes Q^{\mathrm{op}} \arrow[d, "\epsilon_Q"] \\
Q \otimes Q^{\mathrm{op}} \arrow[r, "\epsilon_Q"] & 0\,.
\end{tikzcd}$$

$z \geq x \star y$ iff $x \backslash z \geq y$

$$\begin{tikzcd}
Q^{\mathrm{op}} \otimes Q \otimes Q \arrow[r, "Q^{\mathrm{op}}\otimes\star"] \arrow[d, "-\backslash-\otimes Q"'] & Q^{\mathrm{op}} \otimes Q \arrow[r, "\sigma"] & Q \otimes Q^{\mathrm{op}} \arrow[d, "\epsilon_Q"] \\
Q^{\mathrm{op}} \otimes Q \arrow[r, "\sigma"] & Q \otimes Q^{\mathrm{op}} \arrow[r, "\epsilon_Q"] & 0\,.
\end{tikzcd}$$

# Implications as actions

Let $(A, B)$ be a dual pair such that $(A, \mu_A)$ is a semigroup.
We define $\alpha_A^\ell : A \otimes B \to B$ and $\alpha_A^\rho : B \otimes A \to B$ as the only morphisms such that

$$\begin{tikzcd} A \otimes A \otimes B \arrow[r, "A\otimes\alpha_A^\ell"] \arrow[d, "\mu_A\otimes B"] & A \otimes B \arrow[d, "\epsilon"] \\ A \otimes B \arrow[r, "\epsilon"] & 0 \end{tikzcd} \qquad \begin{tikzcd} B \otimes A \otimes A \arrow[r, "B\otimes\mu_A"] \arrow[d, "\alpha_A^\rho\otimes A"] & B \otimes A \arrow[r, "\sigma"] & A \otimes B \arrow[d, "\epsilon"] \\ B \otimes A \arrow[r, "\sigma"] & A \otimes B \arrow[r, "\epsilon"] & 0\,. \end{tikzcd}$$

Defined that way, $\alpha_A^\rho$ and $\alpha_A^\ell$ are indeed actions, *i.e*

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$$\begin{tikzcd}
A \otimes A \otimes X \arrow[r, shift left, "\mu_A\otimes X"] \arrow[r, shift right, "A\otimes\alpha^\ell"'] & A \otimes X \arrow[r, "\alpha^\ell"] & X .
\end{tikzcd}$$

# Next

# The case of Frobenius quantales

In a Frobenius quantale $(Q, \star, {}^{\perp}(-), (-)^{\perp})$, we have

- $(Q, Q^{\mathrm{op}}, \epsilon)$ is a dual pair;
- $(Q, \star)$ is a semigroup;
- ${}^{\perp}(-), (-)^{\perp} : Q \to Q^{\mathrm{op}}$ and $x \leq {}^{\perp}y$ iff $y \leq x^{\perp}$;

$$x \backslash {}^{\perp}y = x^{\perp}/y$$

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$$\begin{tikzcd}
Q \otimes Q \arrow[r, "A\otimes(-)^\perp"] \arrow[d, "{}^\perp(-)\otimes A"'] & Q \otimes Q^{\mathrm{op}} \arrow[d, "\alpha^\ell_A"] \\
Q^{\mathrm{op}} \otimes Q \arrow[r, "\alpha^\rho_A"'] & Q^{\mathrm{op}}.
\end{tikzcd}$$

# Frobenius structures

**Definition**

A *Frobenius structure* is a tuple $(A, B, \epsilon, \mu_A, l, r)$ where

- $(A, B, \epsilon)$ is a dual pair;
- $(A, \mu_A)$ is a semigroup;
- $l, r : A \longrightarrow B$ and $(l, r)$ is an invertible adjoint pair

such that

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$$\begin{tikzcd} A \otimes A \arrow[r, "A\otimes r"] \arrow[d, "l\otimes A"'] & A \otimes B \arrow[d, "\alpha^{\ell}_A"] \\ B \otimes A \arrow[r, "\alpha^{\rho}_A"'] & B. \end{tikzcd}$$

## Co-multiplication

In a quantale, we can define two comultiplications

$$x \oplus_{\perp} y := {}^{\perp}(y^{\perp} \star x^{\perp}) \qquad x \,{}_{\perp}\!\oplus y := ({}^{\perp}y \star {}^{\perp}x)^{\perp} .$$

In a Frobenius quantale they are actually the same and we have

$$x ⅋ y = {}^{\perp}x \backslash y = x / y^{\perp} .$$

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# The multiplication on $B$

**Proposition**

The diagram on the left commutes iff the diagram on the right does,

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defining a multiplication on $B$.

**Lemma**

1. $(B, \mu_B)$ is a semigroup ;
2. $l$ and $r$ are semigroup morphisms from $(A, \mu_A)$ to $(B, \mu_B)$.
3. $(A, B, \epsilon, \mu_A, l, r)$ is Frobenius iff $(B, A, \epsilon \circ \sigma, \mu_B, r^{-1}, l^{-1})$ is.

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# Frobenius structure and associative bracketed semigroups

**Proposition**

For a Frobenius structure $(A, B, \epsilon, \mu_A, l, r)$, we can define

$$\pi^l_A := \epsilon \circ (A \otimes l) : A \otimes A \to 0\,,$$

We have :

- $(A, \mu_A, \pi^l_A)$ is an associative bracketed semigroup;
- $\pi^l_A$ is a dual pairing.

Conversely, from an associative bracketed semigroup $(A, \mu_A, \pi_A)$ for which $\pi_A$ is a dual pairing, one obtains a Frobenius structure.

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# Frobenius structure and associative bracketed semigroups

**Proposition**

For a Frobenius structure $(A, B, \epsilon, \mu_A, l, r)$, we can define

$$\pi^l_A := \epsilon \circ (A \otimes l) : A \otimes A \to 0 \, ,$$

We have :

- $(A, \mu_A, \pi^l_A)$ is an associative bracketed semigroup;
- $\pi^l_A$ is a dual pairing.

Conversely, from an associative bracketed semigroup $(A, \mu_A, \pi_A)$ for which $\pi_A$ is a dual pairing, one obtains a Frobenius structure.

# Previous work on Frobenius structure

Various work have been done such:

- Lawvere 1969: Frobenius monad;
- Kock 2003: Monoid and comonoid in a monoidal category (same tensor);
- Street 2004: Pseudo-monoid with a pairing $A \otimes A \to I$ making $A$ his own bidual;
- Egger 2010: Monoid and comonoid on a linear distributive category (two different tensor).

# Next

# Nuclearity

From here, $\mathcal{C}$ is symmetric monoidale closed and $0 = I$.

**Definition**

For every object $A$ of $\mathcal{C}$, there exists a canonical arrow

$$\mathtt{mix}_A : A^* \otimes A \longrightarrow [A, A].$$

An object $A$ is *nuclear* if $\mathtt{mix}_A$ is an isomorphism.

**Example**

- In $k$-**Vect** they are the vector spaces of finite dimension.
- In a commutative unital quantale $(Q, \star, 1)$, they are the invertible elements.

**Theorem (Raney 1960, Higgs and Rowe 1989)**

The nuclear objects of **SLatt** are exactly the completely distributive lattices.

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# Adjunction and Nuclearity

**Definition**

For $\eta : I \to B \otimes A$, and $\epsilon : A \otimes B \to I$, $(A, B, \epsilon, \eta)$ is an *adjunction* if

$$\begin{tikzcd}
A \otimes B \otimes A \arrow[d, "\epsilon\otimes A"'] & A \otimes I \arrow[l, "A\otimes\eta"'] \arrow[d, "\rho_A"] \\
I \otimes A \arrow[r, "\ell_A"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes B \arrow[r, "\eta\otimes B"] \arrow[d, "\ell_B"'] & B \otimes A \otimes B \arrow[d, "B\otimes\epsilon"] \\
B & B \otimes I . \arrow[l, "\rho_B"']
\end{tikzcd}$$

**Proposition**

An object is nuclear iff there exist a (right or left) adjoint to it.

# Nuclearity and Frobenius quantale

**Theorem Kruml and Paseka 2008, Santocanale 2020)**

Let $L$ be a complete lattice. The following are equivalent:

- $L$ is a completely distributive lattice.
- The set of endomorphisms of $L$ is a Frobenius quantale.

The first implication is actually a corollary of a more general result.

**Theorem (LS and CL, see last talk)**

Let $L$ be a complete lattice. The image of the Raney's transform $(-)^{\vee} : [L, L]_{\wedge} \to [L, L]$ can always be endowed with a Frobenius quantale structure.

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$$\begin{tikzcd}
L^* \otimes L \cong [L, L]_{\wedge} \arrow[rr, "(-)^{\vee}"] \arrow[dr, two heads] & & {[L, L]} \\
& {[L, L]^t_{\vee}} \arrow[ur, hook] &
\end{tikzcd}$$

# Next

# From Nuclearity to Frobenius structure

**Theorem (LS and CL)**

In a symmetric monoidal closed category, if $A$ is nuclear then $[A, A]$ can be endowed with a Frobenius structure.

**Sketch of the proof**

- We verify that if $\mathtt{mix}$ is invertible, then $(A^* \otimes A, [A, A], \epsilon, \mu_{A^* \otimes A}, \mathtt{mix}, \mathtt{mix})$ is a Frobenius structure.
- As $A^* \otimes A$ is isomorphic to $[A, A]^*$ and Frobenius structures are closed under iso, we obtain the desired theorem.

It has already been noticed that

**Theorem (Street 2004)**

If $X$ has a (right or left) adjoint $X^*$ and $X \cong X^{**}$, then $X^* \otimes X$ is a Frobenius pseudo-monoid.

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# Result

**Theorem (LS and CL)**

Let $\mathcal{C}$ be a $*$-autonomous category such that **Sem**$_{\mathcal{C}}$ has an epi-mono factorization system and $A$ an object of $\mathcal{C}$.
The image of $\mathtt{mix}_A$ can always be endowed with a Frobenius structure.

$$\begin{tikzcd}
A^* \otimes A \arrow[rr, "\mathtt{mix}_A"] \arrow[dr, two heads] & & {[A, A]} \\
& \Im\mathtt{mix}_A \arrow[ur, tail] &
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**Corollary**

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**Conjecture**

Let $([A, A], [A, A]^*, \mu, r, I)$ be a Frobenius structure in an autonomous category. Then $A$ is a nuclear object.

We actually need to add a technical hypothesis.

**Sketch of a proof**

We use the caracterisation of nuclearity with adjoints. So we want:

$$\eta : I \longrightarrow A^* \otimes A \qquad \epsilon : A \otimes A^* \longrightarrow I$$

such that $(A, A^*, \epsilon, \eta)$ is an adjunction.

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such that $(A, A^*, \epsilon, \eta)$ is an adjunction.

## From Frobenius structure to nuclearity

- We identify $[A, A]^*$ with $A^* \otimes A$. Suppose $([A, A], A^* \otimes A, ev, \mu, r, l)$ is a Frobenius structure.
- $[A, A]$ is a monoid. As $r : [A, A] \to A^* \otimes A$ is an iso, $A^* \otimes A$ is also a monoid with unit $\eta : I \to A^* \otimes A$.

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Its composition is given by

$$\begin{tikzcd} A^* \otimes A \otimes A^* \otimes A \arrow[r, "A^* \otimes A \otimes l^{-1}"] \arrow[d, "r^{-1} \otimes A^* \otimes A"'] \arrow[dr, "\mu_{A^* \otimes A}"] & A^* \otimes A \otimes [A, A] \arrow[d, "A^* \otimes ev"] \\ {[A, A] \otimes A^* \otimes A} \arrow[r, "\mu_{A,A,0} \otimes A"'] & A^* \otimes A\,. \end{tikzcd}$$

# From Frobenius structure to nuclearity

That is, we have a diagram of the shape

$$
\begin{tikzcd}
A^* \otimes A \otimes A^* \otimes A \arrow[d, bend right=60, "A^* \otimes g"'] \arrow[d, bend left=60, "h \otimes A"] \\
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# From Frobenius structure to nuclearity

We want:

$$\begin{tikzcd}
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This map actually exits if we ask $I$ to embed into $A$ as a retract, *i.e* if there exists $p : I \to A$ and $c : A \to I$ such that $c \circ p = \mathrm{id}_I$.

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# From Frobenius structure to nuclearity

**Definition**
If for every object $A$ in $\mathcal{C}$, $I$ embeds into $A$ as a retract, $\mathcal{C}$ is *pseudoaffine*.

**Examples**

- **SLatt**
- $k$-**Vect**

**Theorem** (LS and CL)
If $\mathcal{C}$ is pseudoaffine and $([A, A], [A, A]^*, ev, \mu, r, I)$ is a Frobenius structure then $A$ is a nuclear object.

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# Conclusion

## Results

- A definition of Frobenius structures in autonomous categories;
- Generalisation of the double negation construction;
- Proof of our conjecture up to a technical (but quite natural) hypothesis.

## What we will do next

- Connect with linear logic semantic;
- Study the logic of pseudoaffine category;
- Understand "how much" we need $*$-autonomous categories;
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Obrigado pela atenção !

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