On the concept of *Algebraic Crystallography*

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Outline

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A variety $\mathbb{V}$ of UA has a zero object if and only if its theory has a unique constant 0.

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	UA	CT
Jónsson-Tarski	a binary term $+$ such that 0+a=a=a+0	for each pair (X, Y) of objects the pair: $X \overset{l_X}{\rightarrowtail} X \times Y \overset{r_Y}{\leftarrowtail} Y$ is jointly strongly epic

- i.e.: the only subobject of $X \times Y$ containing l_X and r_Y is $1_{X \times Y}$. we then say that the pointed category $\mathbb{E}$ is unital.
- What is rather surprising is the point to which the two different characterizations seem heterogeneous.
- they introduce to very distinct ways of thinking and imagining.

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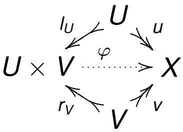
Intrinsic commutative and abelian objects

The unital side introduces a way of thinking an intrinsic commutativity: given any pair $u : U \rightarrowtail X$, $v : V \rightarrowtail X$, there is atmost one factorization making the following diagram commute:



- when such a map does exist, we say that the subobjects u and v commute and call the map φ the *cooperator* of the pair. We denote this situation by $[u, v] = 0$.
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  An object X is commutative when $[1_X, 1_X] = 0$.
- By definition a commutative object X is endowed with a structure $\varphi : X \times X \to X$ of unitary magma which turns out to be an internal commutative monoid.
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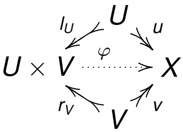

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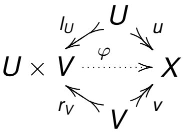

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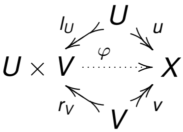

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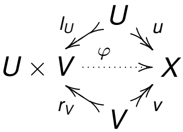

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Whence two intrinsic subcategories:
$\mathit{CoM}(\mathbb{E}) \hookrightarrow \mathbb{E}, \quad \mathit{Ab}(\mathbb{E}) \hookrightarrow \mathbb{E}$
of commutative and abelian objects.

- Introducing this kind of intrinsic notions was the aim of the investigations leading to the notion of unital categories.
- Its varietal origin (Jónsson-Tarski varieties) makes lipid the reason of this intrinsicness: it happens when and because the homomorphism φ coincides with the term $+$
just apply the Eckmann-Hilton argument.
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Of course there is the case where any object X in $\mathbb{E}$ is commutive:

- This is the case if and only if in the unital category $\mathbb{E}$ the canonical map:

$$X + Y \to X \times X$$

is an isomorphism,

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The notion of unital category is a step towards
the notion of Mal’tsev category and the double reading:

Mal’tsev	a ternary term p p(a,b,b)=a p(a,a,b)=b	given any pullback, the pair (s, σ) is jointly strongly epic

- Since a pullback is a ”local product” on the object Y', we get a generalization of unital category:

  a variety/category is a Mal’tsev one if and only if any fibre $Pt_{\mathbb{E}}Y'$ is unital.

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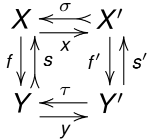
Mal'tsev	a ternary term p p(a,b,b)=a p(a,a,b)=b	given any pullback,  the pair (s, σ) is jointly strongly epic

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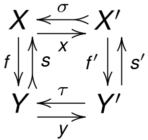
Mal’tsev	a ternary term p p(a,b,b)=a p(a,a,b)=b	given any pullback,  the pair (s, σ) is jointly strongly epic

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Another important notion in UA is the notion of congruence modular variety in which the modular formula for congruences holds:

$$(T \vee S) \wedge R = T \vee (S \wedge R), \text{ for any triple } (T, S, R) \text{ such that} : T \subset R$$

- Gumm (1983) characterized them in geometric terms by the validity of the *Shifting Lemma*:
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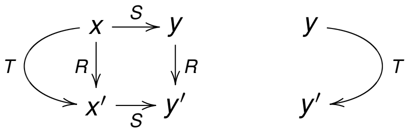

One of the main interest of the Shifting lemma is that, being freed of any condition involving finite colimits, thanks to the Yoneda embedding it keeps a meaning in any finitely complete category $\mathbb{E}$ This led to the notion of Gumm category (B-Gran 2004)

- Once again, this gives rise to a double reading, and a characterization via the fibration of points:

Theorem (B-2005)

Given a category $\mathbb{E}$, the following conditions are equivalent:
1) $\mathbb{E}$ is Gumm category;
2) any fiber $Pt_Y\mathbb{E}$ is congruence hyperextensible.

- **Definition**

A pointed category $\mathbb{E}$ is said to be congruence hyperextensible when given any punctual span and any equivalence relation T on W such that $R[f] \cap R[g] \subset T$, we get $R[f] \cap g^{-1}(t^{-1}(T)) \subset T$.

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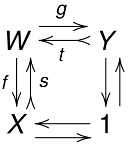
- starting with a punctual span, namely any commutative square of split epimorphism above the zero object: 

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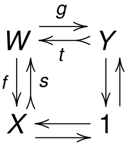

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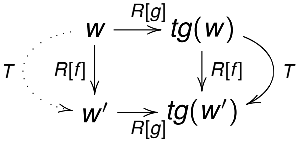

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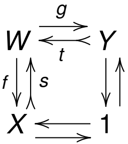
- starting with a punctual span, namely any commutative square of split epimorphism above the zero object: 

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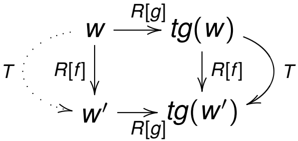
- 

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And here began the surprise.

- because the explanation by the fact that some term in the definition of the variety becomes a homomorphism is no longer valid,
- it cannot remain possible to accept this uniqueness so easily and to keep this uniqueness as an unquestionned process.
- This opens to:
  a new kind of relationship between context and structure.
- So, we propose to call
  crystallographic for a given algebraic structure
  any varietal or categorical setting in which,
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Now, in restrospect, the uniqueness of the autonomous Mal'tsev operations
= affine structure in any congruence modular variety
was actually already noticed by Gumm,

so we can say that any Congruence Modular Variety is crystallographic for the affine structures.

Outline

Gathering all what we have already noticed:

- 1) any pointed Jónnson-Tarski variety or any unital category is crystallographic for the structure of commutative monoid;
- 2) -any stongly unital variety or any strongly unital category

  -any pointed subtractive variety in the sense of Ursini or any subtractive category in the sense of Z. Janelidze

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We can add:
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Outline

An easy first obervation that:
when there is a "duality operator" on the algebraic structure,
the uniqueness property implies that in a crystallographic context
this algebraic structure is necessarily "commutative"
as it is the case for the three first examples above.

- Remark
  Of course, there is the special case where on any object X
  there is one and only one algebraic structure.
- In such a situation we shall speak of
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- linear categories are intensively crystallographic for the notion of commutative monoid;
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So a first question would be:

if an algebraic structure has a non-intensive (let us say extensive) crystallographic context (as it it the case for abelian groups in a CongHyp variety), is there an intensive context (or an extremal crystallographic context relatively to some aspect)?

- And more generally:
  under which conditions a given algebraic structure has an intensive or an extensive crystallographic context?
- Now, on the one hand, this kind of relationship between context and structure has a classical positive side with respect to the context: in that context, any object X has at most one structure;
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Outline

Here is an example of a non-unital CongHyper variety:

$$p_1(a,0,0) = a \qquad p_2(a,0,a) = a, \qquad p_3(0,0,a) = a$$

$$p_1(a,a,b) = p_2(a,a,b)$$

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$$p_i(a,a,a) = a, \quad \forall i \;\; 1 \leq i \leq 3$$

- Let us denote by Hex_3 the variety defined by these only terms and equations.
- We get a fully faithful embedding $h : Gp \to Hex_3$:
  from a group $(G,.)$, construct a Hex_3-algebra on the set G with:
  $p_1(x,y,z) = x.y^{-1}.z; \quad p_2(x,y,z) = z = p_3(x,y,z)$
  We then get a fully faithful restriction $h : Ab \to Ab(Hex_3)$.
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From a K-vector space V, build a Hex_3-algebra on the set V with:
$\bar{p}_1(x, y, z) = x + \frac{-y+z}{2}$, $\bar{p}_2(x, y, z) = \frac{x+z}{2}$, $\bar{p}_3(x, y, z) = \frac{x-y}{2} + z$.

- So, in the variety Hex_3, starting with any K-vector space $\mathbb{V}$, we get the unexpected and remarkable situation where:
- 1) we have two distinct algebras $H(\mathbb{V}, +)$ and $W(\mathbb{V})$ on the same underlying set $\mathbb{V}$
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- So $Ab(Hex_3)$ becomes a very large abelian category containing:
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Consider the congruence modular variety CM_3 defined by the following three ternary terms and equations:

$$p_1(a,b,b) = a \ , \quad p_2(a,b,a) = a \qquad\qquad p_3(b,b,a) = a$$
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- Let *Mal* be the variety defined by a unique ternary operation p satisfying the Mal'tsev identities.
  We get a fully faithful embedding $m : Mal \to CM_3$, constructing a CM_3-algebra on the set X and setting:
  $p_1 = p, \ \ p_2(x,y,z) = z = p_3(x,y,z)$.
- By restriction, we get a fully faithful functor $m : Aff \to Aff(CM_3)$, where *Aff* is the subvariety of *Mal* consisting in its affine objects.

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Now consider any field K with $\chi(K) \neq 2$. We get a faithful functor $a_K : K\text{-}Aff \to Aff(CM_3)$ as well:

- starting from a K-affine space X, construct a CM_3-algebra on the set X by setting:
  $\bar{p}_1(x,y,z) = \beta(\dot{x} + \frac{-\dot{y}+\dot{z}}{2})$, $\bar{p}_2(x,y,z) = \beta(\frac{\dot{x}+\dot{z}}{2})$,
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  where β is the barycentric homomorphism: $\beta : K_1(X) \to X$.
  the affine structure $X \times X \times X \to X$ on the algebra $A_K(X)$ in the variety CM_3 being given by this same $p(x,y,z) = \beta(\dot{x} - \dot{y} + \dot{z})$.
- So $Aff(CM_3)$ becomes a very large finitely complete and cocomplete exact nat. Mal'tsev category containing:
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