Duality theory

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Introduction

A seemingly paradoxical observation

"…an equation is only interesting or useful to the extent that the two sides are different!"

Baez, John and Dolan, James (2001). "From finite sets to Feynman diagrams". In: Mathematics Unlimited – 2001 and Beyond. Ed. by Björn Engquist and Wilfried Schmid. Springer Verlag, pp. 29–50. arXiv: 0004133 [math.QA].

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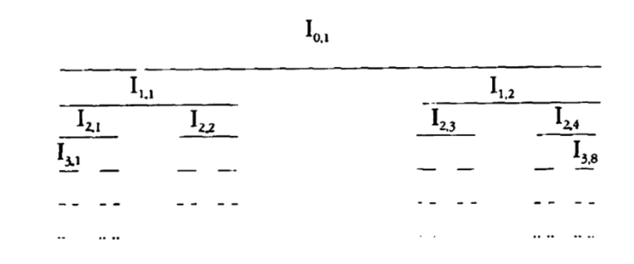

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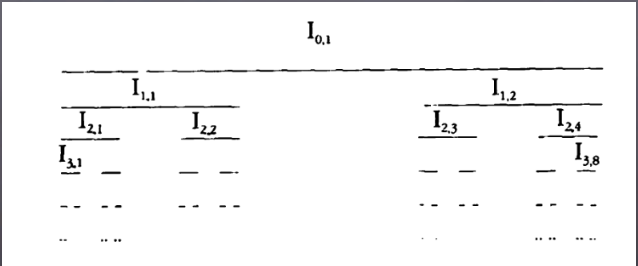

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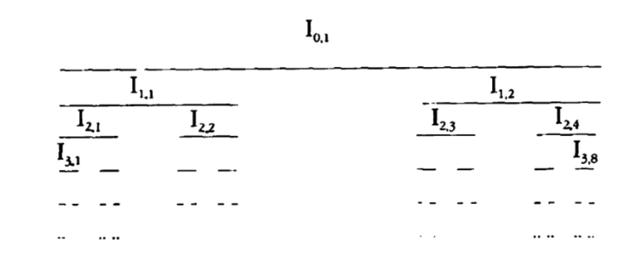

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3. Regarding $\mathsf{CompHausAb}^{\mathrm{op}} \sim \mathsf{Ab}$. An Abelian group is torsion-free if and only if its corresponding compact Hausdorff Abelian group is connected.

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- … does not seem to be easier!!?

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A Kripke model is a triple of the form $\mathcal{C} = (C, \leq, \Vdash)$ where $(C, \leq)$ is a partially ordered set and $\Vdash$ is a binary relation between elements of C and propositional variables so that:

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$$\vDash \varphi \iff \Vdash \varphi.$$

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Sørensen, Morten Heine and Urzyczyn, Pawel (2006). Lectures on the Curry-Howard isomorphism. Vol. 149. Studies in Logic and the Foundations of Mathematics. Elsevier. eprint: https://disi.unitn.it/~bernardi/RSISE11/Papers/curry-howard.pdf.

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- Every finite Heyting algebra is of this form.
- In fact: $\mathsf{Pos}_{\mathrm{fin}}^{\mathrm{op}} \sim \mathsf{HA}_{\mathrm{fin}} \quad (\sim \mathsf{DL}_{\mathrm{fin}})$.

$$\begin{array}{ccc} X & \longmapsto & \mathsf{U}(X) \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle \mathsf{U}(f)} \\ Y & \longmapsto & \mathsf{U}(Y) \end{array} \qquad \begin{array}{ccc} H & \longmapsto & \mathsf{spec}(H) \\ {\scriptstyle g}\downarrow & & \uparrow{\scriptstyle \mathsf{spec}(g)} \\ K & \longmapsto & \mathsf{spec}(K) \end{array}$$

What about the infinite case?

Stone's slogan:
"A cardinal principle of modern mathematical research may be stated as a maxim: One must always topologize."

Stone, Marshall Harvey (1938). "The representation of Boolean algebras". In: Bulletin of the American Mathematical Society 44.(12), pp. 807–816.

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- Spec $\sim$ $\mathrm{DL}^{\mathrm{op}}$ (certain compact spaces vs. distributive lattices).

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Stone, Marshall Harvey (1936). "The theory of representations for Boolean algebras". In: Transactions of the American Mathematical Society 40.(1), pp. 37–111.

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Priestley, Hilary A. (1970). "Representation of distributive lattices by means of ordered Stone spaces". In: Bulletin of the London Mathematical Society 2.(2), pp. 186–190.

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- $\mathsf{EsaSp} \sim \mathsf{HA}^{\mathrm{op}}$ (certain certain ordered spaces vs. Heyting algebras).

Esakia, Leo (1974). "Topological Kripke models". In: Doklady Akademii Nauk SSSR 214, pp. 298–301.

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- EsaSp $\sim$ $\mathsf{HA}^{\mathrm{op}}$ (certain certain ordered spaces vs. Heyting algebras).
- CompHaus $\sim$ $C^*\text{-}\mathsf{Alg}^{\mathrm{op}}$ (compact T2 spaces vs. certain Banach algebras).

Gelfand, Izrail (1941). "Normierte Ringe". In: Recueil Mathématique. Nouvelle Série 9.(1), pp. 3–24.

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  $FX = \mathsf{BooSp}(X, 2)$
  $Ff\colon FY \to FX,\ \varphi \mapsto \varphi \cdot f$
- $G\colon \mathsf{BA}^{\mathrm{op}} \longrightarrow \mathsf{BooSp}$
  $GX = \mathsf{BA}(X, 2)$
  $Gf\colon GY \to GX,\ \psi \mapsto \psi \cdot f$
- $\eta_X\colon X \longrightarrow GFX,$
  $x \longmapsto (\mathrm{ev}_x\colon FX \to 2).$
- $\varepsilon_X\colon X \longrightarrow FGX,$
  $x \longmapsto (\mathrm{ev}_x\colon GX \to 2).$

One more example

Theorem

$$\mathrm{Ab} \sim \mathrm{CompHausAb}^{\mathrm{op}}.$$

Pontrjagin, Lev Semenovich (1934). "The theory of topological commutative groups". In: The Annals of Mathematics 35.(2), p. 361.

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Remark

"That fact is a theorem of topological groups.

Isbell, John R. (1972). "General functorial semantics, I". In: American Journal of Mathematics 94.(2), pp. 535–596.

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"That fact is a theorem of topological groups. That character groups yield an adjoint connection is a theorem of category theory."

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Overview

PART 1: Dual Adjunctions
PART 2: Stone-type dualities
PART 3: Kleisli categories, Splitting idempotents, and all that

Part 1
Dual Adjunctions

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- Lambek, Joachim and Rattray, Basil A. (1979). "A general Stone–Gelfand duality". In: Transactions of the American Mathematical Society 248.(1), pp. 1–35.
- Dimov, Georgi D. and Tholen, Walter (1989). "A characterization of representable dualities". In: Categorical topology and its relation to analysis, algebra and combinatorics: Prague, Czechoslovakia, 22-27 August 1988. Ed. by Ji■í Adámek and Saunders MacLane. World Scientific, pp. 336–357.
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Table of content

1. The structure of dual adjunction

Initial lifts

Definition

Let $F\colon \mathsf{A} \longrightarrow \mathsf{B}$ be a functor. A cone $\mathcal{C} = (f_i\colon C \longrightarrow X_i)_{i\in I}$ in A is said to be initial with respect to F

$$C \qquad\qquad FC \xrightarrow{Ff_i} FX_i$$

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Let $F\colon \mathsf{A} \longrightarrow \mathsf{B}$ be a functor. A cone $\mathcal{C} = (f_i\colon C \longrightarrow X_i)_{i\in I}$ in A is said to be initial with respect to F if for every cone $\mathcal{D} = (g_i\colon D \longrightarrow X_i)_{i\in I}$ and every morphism $h\colon FD \longrightarrow FC$ such that $F\mathcal{D} = F\mathcal{C} \cdot h$

$$\begin{array}{cccc} C & & FC & \xrightarrow{Ff_i} & FX_i \\ & & {\scriptstyle h}\uparrow & \nearrow_{Fg_i} & \\ D & & FD & & \end{array}$$

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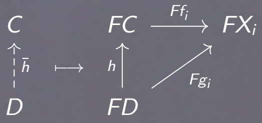

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$$\begin{array}{ccccc} C & & FC & \xrightarrow{Ff_i} & FX_i \\ \uparrow{\scriptstyle \bar{h}} & \longmapsto & {\scriptstyle h}\uparrow & \nearrow{\scriptstyle Fg_i} & \\ D & & FD & & \end{array}$$

Example

- In Top, a cone is initial if and only if the domain has the initial topology.

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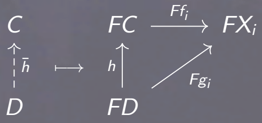

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- In Top, a cone is initial if and only if the domain has the initial topology.
- A cone $(f_i\colon X \longrightarrow X_i)_{i\in I}$ in Ord is initial if and only if, for all $x,y\in X$,

$$x \le y \iff \text{for all } i\in I : f_i(x) \le f_i(y).$$

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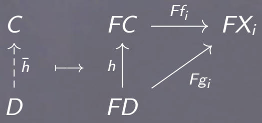

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- In $\mathsf{Grp}, \mathsf{Rng}, \ldots$, every mono-cone is initial.

Initial lifts

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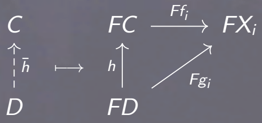

Definition

For a limit preserving faithful functor $|{-}|\colon \mathsf{A} \longrightarrow \mathsf{Set}$, a morphism $m\colon A \longrightarrow B$ in A is an embedding whenever $|m|$ is injective and m is initial.

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Let $F\colon \mathsf{A} \longrightarrow \mathsf{B}$ be a functor. A cone $\mathcal{C} = (f_i\colon C \longrightarrow X_i)_{i \in I}$ in A is said to be initial with respect to F if for every cone $\mathcal{D} = (g_i\colon D \longrightarrow X_i)_{i \in I}$ and every morphism $h\colon FD \longrightarrow FC$ such that $F\mathcal{D} = F\mathcal{C} \cdot h$, there exists a unique A-morphism $\bar{h}\colon D \longrightarrow C$ with $\mathcal{D} = \mathcal{C} \cdot \bar{h}$ and $h = F\bar{h}$.

$$\begin{array}{ccc} C & & \\ \uparrow{\scriptstyle \bar{h}} & \longmapsto & \\ D & & \end{array} \qquad \begin{array}{ccc} FC & \xrightarrow{Ff_i} & FX_i \\ {\scriptstyle h}\uparrow & \nearrow{\scriptstyle Fg_i} & \\ FD & & \end{array}$$

Theorem

Let $F\colon \mathsf{A} \longrightarrow \mathsf{B}$ be a limit preserving faithful functor and $D\colon I \longrightarrow \mathsf{A}$ a diagram. A cone $\mathcal{C}$ for D is a limit of D if and only if the cone $F\mathcal{C}$ is a limit of FD and $\mathcal{C}$ is initial with respect to F.

Initial lifts

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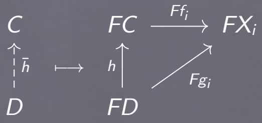
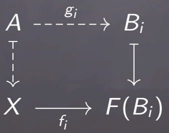


Definition

A functor $F\colon \mathsf{A} \longrightarrow \mathsf{X}$ is topological whenever every cone $(f_i\colon X \longrightarrow UB_i)_{i\in I}$ with a family $(B_i)_{i\in I}$ of A-objects admits an initial lifting, that is, an initial cone $(g_i\colon A \longrightarrow B_i)_{i\in I}$ with $UA = X$ and $Ug_i = f_i$ for all $i \in I$.



Equivalences

Definition

An equivalence between categories A and B consists of functors $F\colon \mathbf{A} \longrightarrow \mathbf{B}$ and $G\colon \mathbf{B} \longrightarrow \mathbf{A}$ together with natural isomorphisms $\eta\colon 1_{\mathbf{A}} \longrightarrow GF$ and $\varepsilon\colon FG \longrightarrow 1_{\mathbf{B}}$.

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We write $\mathsf{A} \sim \mathsf{B}$ if there is an equivalence between A and B.

Equivalences

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An equivalence between categories A and B consists of functors $f\colon A \longrightarrow B$ and $G\colon B \longrightarrow A$ together with natural isomorphisms $\eta\colon 1_A \longrightarrow GF$ and $\varepsilon\colon FG \longrightarrow 1_B$.

We write $A \sim B$ if there is an equivalence between A and B.

Proposition

A functor $F\colon A \longrightarrow B$ is (part of) an equivalence if and only if F is full, faithful and essentially surjective on objects.

Adjunctions

Recall ...

For functors $F\colon \mathsf{A} \longrightarrow \mathsf{B}$ and $G\colon \mathsf{B} \longrightarrow \mathsf{A}$, there is a bijection between

1. pairs of natural transformations $\eta\colon 1_{\mathsf{A}} \longrightarrow GF$ and $\varepsilon\colon FG \longrightarrow 1_{\mathsf{B}}$ satisfying

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\eta_A)} & FGF(A) \\ & {\scriptstyle 1_{F(A)}} \searrow & \downarrow {\scriptstyle \varepsilon_{F(A)}} \\ & & F(A) \end{array} \quad \text{and} \quad \begin{array}{ccc} G(B) & \xrightarrow{\eta_{G(B)}} & GFG(B) \\ & {\scriptstyle 1_{G(B)}} \searrow & \downarrow {\scriptstyle G(\varepsilon_B)} \\ & & G(B) \end{array}$$

   for all A and B, and

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$$\mathsf{B}(F-,-) \longrightarrow \mathsf{A}(-,G-).$$

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An adjunction is a choice of (1) or (2), and we write $F \dashv G$ to indicate that there is an adjunction.

Restricting adjunctions

We consider an adjunction

$$F\colon \mathsf{A} \longrightarrow \mathsf{B}, \quad G\colon \mathsf{B} \longrightarrow \mathsf{A}, \quad \eta\colon 1_{\mathsf{A}} \longrightarrow GF, \quad \varepsilon\colon FG \longrightarrow 1_{\mathsf{B}}, \qquad (*)$$

and the full subcategories

$$\mathrm{Fix}(\eta) \quad \text{and} \quad \mathrm{Fix}(\varepsilon)$$

of A (resp. B) defined by all objects A in A (resp. B in B) where η_A (resp. ε_B) is an isomorphism.

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Theorem

1. The adjunction $(*)$ restricts to an equivalence $\mathrm{Fix}(\eta) \sim \mathrm{Fix}(\varepsilon)$.
2. The following assertions are equivalent.
   (i) $\mathrm{Fix}(\eta) \hookrightarrow \mathsf{A}$ is right adjoint with left adjoint GF (the monad $(GF, \eta, G\varepsilon_F)$ is idempotent).
   (ii) η_G is an isomorphism.
   (iii) $\mathrm{Fix}(\varepsilon) \hookrightarrow \mathsf{A}$ is left adjoint with right adjoint FG.
   (iv) ε_G is an isomorphism.

Dual adjunctions

Notation

In the sequel we typically consider adjunctions

$$F\colon \mathsf{A} \longrightarrow \mathsf{B}^{\mathrm{op}}, \quad G\colon \mathsf{B}^{\mathrm{op}} \longrightarrow \mathsf{A}, \quad \eta\colon 1_{\mathsf{A}} \longrightarrow GF, \quad \varepsilon\colon FG \longrightarrow 1_{\mathsf{B}^{\mathrm{op}}},$$

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$$\begin{array}{ccc} F(A) & \xrightarrow{\varepsilon_{F(A)}} & FGF(A) \\ & {}_{1_{F(A)}}\searrow & \downarrow{\scriptstyle F(\eta_A)} \\ & & F(A) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G(B) & \xrightarrow{\eta_{G(B)}} & GFG(B) \\ & {}_{1_{G(B)}}\searrow & \downarrow{\scriptstyle G(\varepsilon_B)} \\ & & G(B) \end{array}$$

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Example

For a category A with an object $\widetilde{A}$ with arbitrary powers, we have the adjunction defined by

$$\mathsf{A}(-,\widetilde{A})\colon \mathsf{A}^{\mathrm{op}} \longrightarrow \mathsf{Set} \qquad \widetilde{A}^{(-)}\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{A}$$

$$\eta_A\colon A \longrightarrow \widetilde{A}^{\mathsf{A}(A,\widetilde{A})} \qquad \varepsilon_X\colon X \longrightarrow \mathsf{A}(\widetilde{A}^X,\widetilde{A}).$$

Dual adjunctions come from dualising objects

Theorem
Assume that concrete categories (A, U) and (B, V) with $U \simeq \mathsf{A}(A_0, -)$ and $V \simeq \mathsf{B}(B_0, -)$ and a dual adjunction

$$F\colon \mathsf{A} \longrightarrow \mathsf{B}^{\mathrm{op}}, \quad G\colon \mathsf{B}^{\mathrm{op}} \longrightarrow \mathsf{A}, \quad \eta\colon 1_{\mathsf{A}} \longrightarrow GF, \quad \varepsilon\colon 1_{\mathsf{B}} \longrightarrow FG$$

are given. Put $\widetilde{A} = F(B_0)$ and $\widetilde{B} = G(A_0)$. Then the following assertions hold.

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Remark
We say that the adjunction is represented by $(\widetilde{A}, \widetilde{B})$.

Units are evaluation

We assume now

$$VF = \mathsf{A}(-, \tilde{A}) \quad \text{and} \quad UG = \mathsf{B}(-, \tilde{B})$$

and consider the "evaluation maps" (writing $U = |-| = V$)

$$\begin{aligned} \mathrm{ev}_{A,a} \colon \mathsf{A}(A, \widetilde{A}) = |FA| &\longrightarrow |\widetilde{A}| \\ \varphi &\longmapsto |\varphi|(a) \end{aligned}$$

and

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and

$$|\widetilde{A}| \xrightarrow{|\eta_{\widetilde{A}}|} |GF(\widetilde{A})| \xrightarrow{\mathsf{ev}_{F(\widetilde{A}),\mathbf{1}_{\widetilde{A}}}} |\widetilde{B}|, \qquad |\widetilde{B}| \xrightarrow{|\varepsilon_{\widetilde{B}}|} |FG(\widetilde{B})| \xrightarrow{\mathsf{ev}_{G(\widetilde{B}),\mathbf{1}_{\widetilde{B}}}} |\widetilde{B}|.$$

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(with the composites labelled τ and σ respectively)

Theorem

$$\tau \cdot \mathrm{ev}_{A,a} = |\eta_A|(a), \quad \sigma \cdot \mathrm{ev}_{B,b} = |\varepsilon_B|(b), \quad \tau = \sigma^{-1}.$$

Units are evaluation

Proof.
About the first affirmation. For $\varphi \colon A \longrightarrow \widetilde{A}$:

$$\tau \cdot \mathrm{ev}_{A,a}(\varphi) = ev_{F(\widetilde{A},1_{\widetilde{A}})} \cdot |\eta_{\widetilde{A}}| \cdot \mathrm{ev}_{A,a}(\varphi)$$

□

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$$\begin{aligned}\tau \cdot \mathsf{ev}_{A,a}(\varphi) &= ev_{F(\widetilde{A},1_{\widetilde{A}})} \cdot |\eta_{\widetilde{A}}| \cdot \mathsf{ev}_{A,a}(\varphi)\\ &= (|\eta_{\widetilde{A}}| \cdot \mathsf{ev}_{A,a}(\varphi))(1_{\widetilde{A}})\\ &= (|\eta_{\widetilde{A}} \cdot \varphi|(a))(1_{\widetilde{A}})\end{aligned}$$

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Summing up

For concrete categories $(\mathsf{A}, |-|)$ and $(\mathsf{B}, |-|)$ with representable forgetful functors and a dual adjunction

$$F\colon \mathsf{A} \longrightarrow \mathsf{B}^{\mathrm{op}}, \quad G\colon \mathsf{B}^{\mathrm{op}} \longrightarrow \mathsf{A}, \quad \eta\colon 1_{\mathsf{A}} \longrightarrow GF, \quad \varepsilon\colon 1_{\mathsf{B}} \longrightarrow FG,$$

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$$|F| = \mathsf{A}(-, \widetilde{A}), \quad |G| = \mathsf{B}(-, \widetilde{B}), \quad |\eta_A|(a) = \mathrm{ev}_{A,a}, \quad |\varepsilon_B|(B) = \mathrm{ev}_{B,b}\,.$$

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Remark

We have

$$\begin{array}{ccc} A & \xrightarrow{|\eta_X|} & \mathsf{B}(FA, \widetilde{B}), \\ & \searrow^{|f|} & \big\downarrow \mathrm{ev}_f \\ & & \widetilde{A} \end{array} \qquad \begin{array}{ccc} a & \longmapsto & \mathrm{ev}_{A,a} \\ & \searrow & \big\downarrow \\ & & f(a). \end{array}$$

Therefore:

$$\eta_A \text{ is mono} \iff (f\colon A \longrightarrow \widetilde{A})_f \text{ is mono.}$$

Regular cogenerators

Remark

Assume that $\widetilde{C}$ is a regular cogenerator in a category $\mathcal{C}$ with arbitrary powers of $\widetilde{C}$. It follows that, for each object C in $\mathcal{C}$, there exists an equalizer diagram

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Assume that $\widetilde{C}$ is a regular cogenerator in a category C with arbitrary powers of $\widetilde{C}$. It follows that, for each object C in C, there exists an equalizer diagram

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Hence, a right adjoint, full and faithful functor $F\colon \mathsf{B} \longrightarrow \mathsf{C}$ is an equivalence provided that $\widetilde{C}$ is, up to isomorphism, contained in the image of F.

2. How to construct dual adjunctions

Dualising objects

How can we construct a dual adjunction between given concrete categories $(A, |-|)$ and $(B, |-|)$ over Set?

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1. for each object A in A, the cone

$$(\mathrm{ev}_{A,a} \colon \mathsf{A}(A, \widetilde{A}) \longrightarrow |\widetilde{B}|)_{a \in |A|}$$

admits a lifting

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3. for each object A in A, the map

$$\begin{aligned} \eta_A \colon |A| &\longrightarrow |GF(A)| = \mathsf{B}(F(A), \widetilde{B}) \\ a &\longmapsto \mathrm{ev}_{A,a} \end{aligned}$$

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How to guarantee this?

Theorem

If the following two conditions are satisfied:

(A) For each object A in A, the cone

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(B) For each object B in B, the cone

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then $(\widetilde{A}, \widetilde{B})$ induce a (natural) dual adjunction.

And how to get this?

Proposition

1. If $|-|: A \longrightarrow \mathrm{Set}$ is topological, then (A).

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   2.1 all powers of $\widetilde{A}$ exist in A and are preserved by $|-|\colon \mathsf{A} \longrightarrow \mathsf{Set}$, and
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   Then (B).

Proof.

Let A be an object of A and θ be an operation symbol with arity n. We define

$$\mathsf{A}(A, \widetilde{A})^n \longrightarrow \mathsf{A}(A, \widetilde{A}), \quad (h_i)_i \longmapsto (A \xrightarrow{\langle h_i \rangle} \widetilde{A}^n \xrightarrow{\theta^{\widetilde{B}}} \widetilde{A}).$$

Then put $F(A) = (\mathsf{A}(A, \widetilde{A}), \ldots$ these operations $\ldots)$; hence $F(A)$ is a subalgebra of $\widetilde{B}^{|A|}$. □

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   Then (B). If, moreover, A is concretely $\widetilde{A}$-complete, then also (A).

Definition

The category A is concretely $\widetilde{A}$-complete if all powers of $\widetilde{A}$ and all equalisers of pairs of parallel maps between powers of $\widetilde{A}$ exist in A, and these limits are preserved by $|-|\colon \mathsf{A} \longrightarrow \mathsf{Set}$.

Proof of the last affirmation

A map $f\colon |B| \longrightarrow |\widetilde{B}|$ is an algebra homomorphism if and only if, for every operation symbol θ (with arity n), the diagram

$$\begin{array}{ccc} |B|^n & \xrightarrow{\ f^n\ } & |\widetilde{B}|^n \\ {\scriptstyle \theta^B}\big\downarrow & & \big\downarrow{\scriptstyle \theta^{\widetilde{B}}} \\ |B| & \xrightarrow[\ f\ ]{} & |\widetilde{B}| \end{array}$$

commutes, that is: $f \cdot \theta^B(h) = \theta^{\widetilde{B}} \cdot f^n(h)$ for all $h \in |B|^n$.

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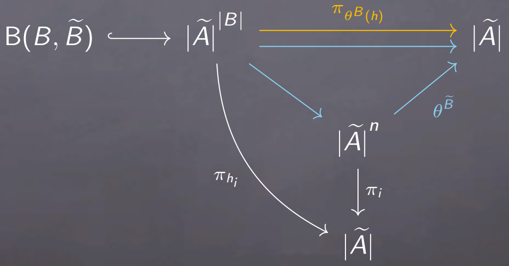

Initial cogenerators

Remark

We consider a natural dual adjunction

$$F\colon \mathsf{A} \longrightarrow \mathsf{B}^{\mathrm{op}}, \quad G\colon \mathsf{B}^{\mathrm{op}} \longrightarrow \mathsf{A}, \quad \eta\colon 1_{\mathsf{A}} \longrightarrow GF, \quad \varepsilon\colon 1_{\mathsf{B}} \longrightarrow FG \qquad (*)$$

induced by $\widetilde{A}$ and $\widetilde{B}$. Then

$$\eta_A \text{ is an embedding} \iff (f\colon A \to \widetilde{A})_f \text{ is point-separating and initial.}$$

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Definition

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Remark

The adjunction $(*)$ restricts to the full subcategories $\mathsf{InitCog}(\widetilde{A})$ and $\mathsf{InitCog}(\widetilde{B})$ "initially cogenerated by $\widetilde{A}$ and $\widetilde{B}$".

3. Gelfand-duality

C^*-algebras

Definition

A C^*-algebra is a commutative unital $\mathbb{C}$-algebra with norm $\|-\|$ and involution $(-)^*$ which is complete with respect to $\|-\|$ and satisfies (besides the "expected" axioms)

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Example

For each topological space X,

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Proposition

For each C^*-algebra B and each element $x \in B$,

$$\|x\| = \sup\{|\varphi(x)| \mid \varphi \in C^*\text{-Alg}(B, \mathbb{C})\}.$$

Gelfand, Izrail (1941). "Normierte Ringe". In: Recueil Mathématique. Nouvelle Série 9.(1), pp. 3–24.

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Remark

Hence, every homomorphism of C^*-algebras satisfies $\|f(x)\| \leq \|x\|$ and $\mathbb{C}$ is a cogenerator in C^*-Alg.

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We consider the unit-ball functor $|-| = \bigcirc \colon C^*\text{-Alg} \longrightarrow \mathrm{Set}$.

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The functor above is even monadic.

Negrepontis, Joan Wick (1971). "Duality in analysis from the point of view of triples". In: Journal of Algebra 19.(2), pp. 228–253.

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$$(\mathrm{ev}_{X,a} : \mathsf{Top}(X, \mathbb{D}) \longrightarrow \mathbb{D} = |\mathbb{C}|)_{x \in X}.$$

Corollary

The pair $(\mathbb{D}, \mathbb{C})$ induce a natural dual adjunction

$$C^* \colon \mathsf{Top}^{\mathrm{op}} \longrightarrow C^*\text{-Alg}, \quad S \colon C^*\text{-Alg} \longrightarrow \mathsf{Top}^{\mathrm{op}}.$$

The concrete category C^*-Alg

We consider the unit-ball functor $|-| = \bigcirc\colon C^*\text{-Alg} \longrightarrow \mathsf{Set}$.

Remark

The functor above is even monadic.

Remark

For a topological space X,

$$C^*(X) = \{f\colon X \longrightarrow \mathbb{C} \quad \text{continuous and bounded}\}$$

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$$C^*\colon \mathsf{Top}^{\mathrm{op}} \longrightarrow C^*\text{-Alg}, \quad S\colon C^*\text{-Alg} \longrightarrow \mathsf{Top}^{\mathrm{op}}.$$

Remark

For each C^*-algebra B, the space $S(B)$ is compact Hausdorff (being en equaliser of a pair of continuous maps between powers of $\mathbb{D}$).

Cogenerator properties

Proposition

η_X is an embedding if and only of X is completely regular.

Cogenerator properties

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Proposition

For each C^*-algebra B, ε_B is an embedding.

Obtaining the equivalence

Theorem (Stone-Weierstrass)

Let A be a compact Hausdorff space and let $M \subseteq C^*(A)$ be a C^*-subalgebra of $C^*(A)$ such that the cone $(f \colon A \longrightarrow \mathbb{D})_{f \in \bigcirc(M)}$ separates the points of A. Then $M = C^*(A)$.

Obtaining the equivalence

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Corollary

For every C^*-algebra B, $\varepsilon_B\colon B \longrightarrow C^*(S(B))$ is surjective.

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For every C^*-algebra B, $\varepsilon_B\colon B \longrightarrow C^*(S(B))$ is surjective.

Theorem

Let B be a C^*-algebra and let $M \subseteq S(B)$ be a closed subspace of $S(B)$ such that the cone $(f\colon B \longrightarrow \mathbb{C})_{f \in M}$ separates the points of B. Then $M = S(B)$.

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Corollary

For every compact Haudorff space A, $\eta_A\colon A \longrightarrow S(C^*(A))$ is surjective.

Obtaining the equivalence

Theorem (Stone-Weierstrass)

Let A be a compact Hausdorff space and let $M \subseteq C^*(A)$ be a C^*-subalgebra of $C^*(A)$ such that the cone $(f\colon A \longrightarrow \mathbb{D})_{f \in \bigcirc(M)}$ separates the points of A. Then $M = C^*(A)$.

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For every C^*-algebra B, $\varepsilon_B\colon B \longrightarrow C^*(S(B))$ is surjective.

Theorem

Let B be a C^*-algebra and let $M \subseteq S(B)$ be a closed subspace of $S(B)$ such that the cone $(f\colon B \longrightarrow \mathbb{C})_{f \in M}$ separates the points of B. Then $M = S(B)$.

Corollary

For every compact Haudorff space A, $\eta_A\colon A \longrightarrow S(C^*(A))$ is surjective.

Theorem

$\mathrm{CompHaus}^{\mathrm{op}} \sim C^*\text{-}\mathrm{Alg}$ (and $\mathrm{CompHaus} \hookrightarrow \mathrm{Top}$ is reflective).

Some history

- $\mathsf{CompHaus}^{\mathrm{op}} \xrightarrow{\hom(-,[0,1])} \mathsf{Set}$ is monadic.

Duskin, John (1969). "Variations on Beck's tripleability criterion". In: Reports of the Midwest Category Seminar III. Ed. by Saunders MacLane. Springer Berlin Heidelberg, pp. 74–129.

Some history

- $\mathsf{CompHaus}^{\mathrm{op}} \xrightarrow{\hom(-,[0,1])} \mathsf{Set}$ is monadic.
- $[0,1]$ is $\aleph_1$-copresentable in CompHaus.

Gabriel, Peter and Ulmer, Friedrich (1971). Lokal präsentierbare Kategorien. Vol. 221. Lecture Notes in Mathematics. Berlin: Springer-Verlag. v + 200.

Some history

- $\mathsf{CompHaus}^{\mathrm{op}} \xrightarrow{\hom(-,[0,1])} \mathsf{Set}$ is monadic.
- $[0,1]$ is $\aleph_1$-copresentable in $\mathsf{CompHaus}$.
- The algebraic theory of $\mathsf{CompHaus}^{\mathrm{op}}$ can be generated by 5 operations.

Isbell, John R. (1982). "Generating the algebraic theory of $C(X)$". In: Algebra Universalis 15.(2), pp. 153–155.

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- $\mathsf{CompHaus}^{\mathrm{op}} \xrightarrow{\hom(-,[0,1])} \mathsf{Set}$ is monadic.
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Marra, Vincenzo and Reggio, Luca (2017). "Stone duality above dimension zero: Axiomatising the algebraic theory of $C(X)$". In: Advances in Mathematics 307, pp. 253–287. arXiv: 1508.07750 [math.LO].

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- Similarly, $\mathsf{PosComp}^{\mathrm{op}}$ is a quasivariety.

Hofmann, Dirk, Neves, Renato, and Nora, Pedro (2018). "Generating the algebraic theory of $C(X)$: the case of partially ordered compact spaces". In: Theory and Applications of Categories 33.(12), pp. 276–295. arXiv: 1706.05292 [math.CT].

Some history

- $\mathsf{CompHaus}^{\mathrm{op}} \xrightarrow{\hom(-,[0,1])} \mathsf{Set}$ is monadic.
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- Similarly, $\mathsf{PosComp}^{\mathrm{op}}$ is a quasivariety.
- Even better, $\mathsf{PosComp}^{\mathrm{op}}$ is a variety.

Abbadini, Marco (2021). "On the axiomatisability of the dual of compact ordered spaces". PhD thesis. Università degli Studi di Milano.

Abbadini, Marco and Reggio, Luca (2020). "On the axiomatisability of the dual of compact ordered spaces". In: Applied Categorical Structures 28.(6), pp. 921–934. arXiv: 1909.01631 [math.CT].

4. Stone-Weierstraß condition

The setting

Let C be a complete category and let $\mathbb{M}$ be a class of C-morphisms satisfying the following conditions:

1. RegMono(C) $\subseteq \mathbb{M} \subset$ Mono(C),
2. $\mathbb{M}$ is closed under composition, stable under pullbacks and
3. for each family $(m_i\colon A_i \longrightarrow A)_{i\in I}$ of $\mathbb{M}$-morphisms, there exist an intersection $d\colon D \longrightarrow A$ and $d \in \mathbb{M}$.

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$\mathbb{M} = \{$embeddings$\}$ or $\mathbb{M} = \{$regular monos$\}$.

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Examples

$\mathbb{M} = \{$embeddings$\}$ or $\mathbb{M} = \{$regular monos$\}$.

Remark

$\mathbb{M}$ is part of a factorization structure ($\mathbb{M}$-ExtrEpi, $\mathbb{M}$) for morphisms in C.

Adámek, Ji■í, Herrlich, Horst, and Strecker, George E. (1990). Abstract and concrete categories: The joy of cats. Pure and Applied Mathematics (New York). New York: John Wiley & Sons Inc. xiv + 482. Republished in: Reprints in Theory and Applications of Categories, No. 17 (2006) pp. 1–507.

Some notation

We define the following class of small cones of C:

$$\mathcal{M} = \{(f_i \colon C \longrightarrow C_i)_{i \in I} \mid I \text{ is a set and } \langle f_i \rangle_{i \in I} \in \mathbb{M}\}.$$

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Remark

Each limit cone belongs to $\mathcal{M}$ and a small cone belongs to $\mathcal{M}$ if and only if it contains a $\mathcal{M}$-cone.

$\mathcal{M}$ is closed under composition if and only if $\mathbb{M}$ is stable under products.

Some notation

We define the following class of small cones of $\mathbb{C}$:

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Each limit cone belongs to $\mathcal{M}$ and a small cone belongs to $\mathcal{M}$ if and only if it contains a $\mathcal{M}$-cone.

$\mathcal{M}$ is closed under composition if and only if $\mathbb{M}$ is stable under products.

Definition

Let $\widetilde{C}$ be a $\mathbb{C}$-object. $\widetilde{C}$ is called an $\mathbb{M}$-cogenerator of $\mathbb{C}$ if, for each object C in $\mathbb{C}$, the cone $(f \colon C \longrightarrow \widetilde{C})_f$ belongs to $\mathcal{M}$.

More setting

We consider a dual adjunction

$$F\colon \mathsf{A} \longrightarrow \mathsf{B}^{\mathrm{op}}, \quad G\colon \mathsf{B}^{\mathrm{op}} \longrightarrow \mathsf{A}, \quad \eta\colon 1_{\mathsf{A}} \longrightarrow GF, \quad \varepsilon\colon 1_{\mathsf{B}} \longrightarrow FG$$

induced by $\widetilde{A}$ and $\widetilde{B}$.

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induced by $\widetilde{A}$ and $\widetilde{B}$.

Furthermore, there are classes $\mathbb{M}_{\mathsf{A}}$ and $\mathbb{M}_{\mathsf{B}}$ of A-morphisms resp. B-morphisms satisfying ... (see before) ... and so that the cones

$$(\mathrm{ev}_{A,a}\colon G(A) \longrightarrow \widetilde{B})_{a\in A} \quad \text{and} \quad (\mathrm{ev}_{B,b}\colon F(B) \longrightarrow \widetilde{A})_{b\in B}$$

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belong to $\mathcal{M}_{\mathsf{B}}$ resp. $\mathcal{M}_{\mathsf{A}}$.

Finally, $\widetilde{A}$ is a $\mathbb{M}_{\mathsf{A}}$-cogenerator of A and $\widetilde{B}$ is a $\mathbb{M}_{\mathsf{B}}$-cogenerator of B.

Injectivity

Assume that our given adjunction is already and equivalence.

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Proposition

1. The following are equivalent.
   1.1 $F(\mathbb{M}_A) \subseteq \mathbb{M}_B$-ExtrEpi.
   1.2 $G(\mathbb{M}_B) \subseteq \mathbb{M}_A$-ExtrEpi.
2. The following are equivalent.
   2.1 $F(\mathbb{M}_A\text{-ExtrEpi}) \subseteq \mathbb{M}_B$.
   2.2 $G(\mathbb{M}_B\text{-ExtrEpi}) \subseteq \mathbb{M}_A$.

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   2.1 $F(\mathbb{M}_{\mathsf{A}}\text{-ExtrEpi}) \subseteq \mathbb{M}_{\mathsf{B}}$.
   2.2 $G(\mathbb{M}_{\mathsf{B}}\text{-ExtrEpi}) \subseteq \mathbb{M}_{\mathsf{A}}$.

Remark

If

$$\mathbb{M}_{\mathsf{B}}\text{-ExtrEpi} = \{\text{Surjections}\} = \mathbb{M}_{\mathsf{A}}\text{-ExtrEpi}$$

then $\widetilde{A}$ is $\mathbb{M}_{\mathsf{A}}$-injective if and only if $\widetilde{B}$ is $\mathbb{M}_{\mathsf{B}}$-injective.

The Stone-Weierstraß condition

Definition

F satisfies the Stone-Weierstraß condition provided that

(SW) For each object A in A, a $\mathbb{M}_{\mathrm{B}}$-morphism $m\colon M \longrightarrow F(A)$ is an isomorphism provided that the cone $(m(f)\colon A \longrightarrow \widetilde{A})_{f \in M} \in \mathcal{M}_{\mathrm{A}}$.

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Proposition

If F satisfies (SW) then $F(\mathbb{M}_{\mathsf{A}}) \subseteq \mathbb{M}_{\mathsf{B}}$-ExtrEpi.

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Proposition

If F satisfies (SW) then $F(\mathbb{M}_{\mathsf{A}}) \subseteq \mathbb{M}_{\mathsf{B}}$-ExtrEpi.

Proposition

Assume that our dual adjunction is a dual equivalence and $F(\mathbb{M}_{\mathsf{A}}) \subseteq \mathbb{M}_{\mathsf{B}}$-ExtrEpi. Then F satisfies (SW).

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Proposition

Assume that our dual adjunction is a dual equivalence and $F(\mathbb{M}_{\mathsf{A}}) \subseteq \mathbb{M}_{\mathsf{B}}$-ExtrEpi. Then F satisfies (SW).

Corollary

If we have a dual equivalence, G satisfies (SW) if and only if F satisfies (SW).

The clone condition

Definition

F satisfies the clone-condition provided that the following holds:

(Cl) For each set X, every $\mathbb{M}_{\mathrm{B}}$-morphism $m\colon M \longrightarrow F(\widetilde{A}^X)$ is an isomorphism provided that the cone $(m(f)\colon \widetilde{A}^X \longrightarrow \widetilde{A})_{f\in|M|}$ contains all projections.

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Remark

If B is a category of algebras, then the condition above means that

$$|\mathrm{Clone}_X(\widetilde{B})| = |\mathsf{A}(\widetilde{A}^X, \widetilde{A})|.$$

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If B is a category of algebras, then the condition above means that

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Proposition

If the given dual adjunction is an equivalence, then F satisfies (Cl).

Relation with Stone-Weierstrass

Proposition

If F satisfies (CI) and $F(\mathbb{M}_{\mathrm{A}}) \subseteq \mathbb{M}_{\mathrm{B}}$-ExtrEpi, then F satisfies (SW).

Relation with Stone-Weierstrass

Proposition

If F satisfies (CI) and $F(\mathbb{M}_{\mathsf{A}}) \subseteq \mathbb{M}_{\mathsf{B}}$-ExtrEpi, then F satisfies (SW).

Theorem

Assume that B is the category of Σ-algebras and homomorphisms (for a signature Σ), here $\mathbb{M}_{\mathsf{B}} = \{\text{monos}\}$ and $\mathbb{M}_{\mathsf{A}} = \{\text{regular monos}\}$. Then the following assertions are equivalent

(i) The dual adjunction is an equivalence.

(ii) The following three conditions are fulfilled.

   (a) A is concretely $\widetilde{A}$-complete.

   (b) $\widetilde{A}$ is a regular injective regular cogenerator of A.

   (c) For each set X,

$$|\mathrm{Clone}_X(\widetilde{B})| = |\mathsf{A}(\widetilde{A}^X, \widetilde{A})|.$$

Part 2
Stone-type dualities

Some references

- Clark, David M. and Davey, Brian A. (1998). Natural dualities for the working algebraist. Vol. 57. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press. xii + 356.
- Johnstone, Peter T. (1986). Stone spaces. Vol. 3. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press. xxii + 370. Reprint of the 1982 edition.

The idea

Let C and D be small categories. If

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Let C and D be small categories. If

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Let C and D be small categories. If

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Ind(C) is the free cocompletion of C under filtered colimits.

Pro(D) is the free cocompletion of D under cofiltered limits.

Our strategy

We consider a dual adjunction

$$F\colon \mathsf{A} \longrightarrow \mathsf{B}^{\mathrm{op}}, \quad G\colon \mathsf{B}^{\mathrm{op}} \longrightarrow \mathsf{A}, \quad \eta\colon 1_{\mathsf{A}} \longrightarrow GF, \quad \varepsilon\colon 1_{\mathsf{B}} \longrightarrow FG \qquad (*)$$

induced by $\widetilde{A}$ and $\widetilde{B}$.

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Then the adjunction $(*)$ is actually an equivalence provided that

- Each object B in B is a filtered colimit of finite objects.
- F sends cofiltered limits of finite objects to colimits.
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Remark

Under the conditions above, the endofunctor $FG\colon \mathsf{B} \longrightarrow \mathsf{B}$ preserves filtered colimits of finite objects and, dually, $GF\colon \mathsf{A} \longrightarrow \mathsf{A}$ preserves cofiltered limits of finite objects.

Table of content

5. Locally presentable categories

Limit sketches

Definition

A finitary limit sketch is a triple $\mathcal{S} = (\mathrm{C}, \mathcal{L}, \sigma)$ consisting of

- a small category C,
- a set $\mathcal{L}$ of diagrams in C with finite shape, and
- a function σ which assigns to each diagram of $\mathcal{L}$ a cone.

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A model of a finitary limit sketch $\mathcal{S} = (\mathsf{C}, \mathcal{L}, \sigma)$ in a category A is a functor $M\colon \mathsf{C} \longrightarrow \mathsf{A}$ which sends each diagram $D\colon I \longrightarrow \mathsf{C}$ of $\mathcal{L}$ to a limit $\sigma(D)$ of FD.

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Finally, $\mathsf{Mod}(\mathcal{S}, \mathsf{A})$ denotes the full subcategory of the functor category $\mathsf{A}^{\mathcal{C}}$ defined by all models of $\mathcal{S}$ in A.

Remark

$\mathsf{Mod}(\mathcal{S}, \mathsf{A})$ is reflective in $\mathsf{A}^{\mathcal{C}}$.

Kennison, John F. (1968). "On limit-preserving functors". In: Illinois Journal of Mathematics 12.(4), pp. 616–619.

Freyd, P. J. and Kelly, G. M. (1972). "Categories of continuous functors, I". In: Journal of Pure and Applied Algebra 2.(3), pp. 169–191.

Example

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Then $\mathrm{Mod}(\mathcal{S}, \mathrm{Set})$ is the category of magmas and magma homomorphisms

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Then $\mathsf{Mod}(\mathcal{S}, \mathsf{Set})$ is the category of magmas and magma homomorphisms, $\mathsf{Mod}(\mathcal{S}, \mathsf{CompHaus})$ is the category of "compact Hausdorff magmas" and ...

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$$\begin{array}{ccc} & c_2 & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ c_1 & & c_1 \end{array} \qquad \begin{array}{ccc} r & \xrightarrow{1_r} & r \\ {\scriptstyle 1_r}\downarrow & & \downarrow{\scriptstyle m} \\ r & \xrightarrow[m]{} & c_2 \end{array}$$

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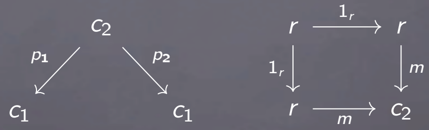

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Then $\mathrm{Mod}(\mathcal{S}, \mathrm{Set})$ is category of sets equipped with a binary relation and relation-preserving maps, ...

And still one more example

For a finitely complete small category C, we may consider the limit sketch $\mathcal{S} = (C, \mathcal{L}, \sigma)$ where

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Then $\mathrm{Mod}(\mathcal{S}, \mathrm{Set}) \sim \mathrm{Cart}(\mathrm{C}, \mathrm{Set})$.

Locally presentable categories

Definition

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Locally finitely presentable categories are also complete, (co)wellpowered and have a generating set. Moreover, each functor between locally finitely presentable categories which preserves limits and filtered colimits has a left adjoint.

Gabriel and Ulmer (1971)

The model categories of finitary limit sketches in Set are precisely (up to equivalence) the locally finitely presentable categories. More precisely, $(\mathrm{Set}, \mathrm{Set})$ represent a dual equivalence

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- Gabriel, Peter and Ulmer, Friedrich (1971). Lokal präsentierbare Kategorien. Vol. 221. Lecture Notes in Mathematics. Berlin: Springer-Verlag. v + 200.
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- The category of models of a limit sketch in a locally presentable category is locally presentable.
- The category of models of a colimit sketch in a locally presentable category is locally presentable.

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- For an object C in C, we define a chain $\mathcal{G}_n(C)$ ($n \in \mathbb{N}$) of full subcategories of C in the following way:
  1. We put $\mathcal{G}_0(C) = \{C\}$ and,
  2. for each $n \geq 0$, $\mathcal{G}_{n+1}(C) = \mathrm{Sub}_{\mathcal{S}}(\mathrm{Lim}_{\mathcal{S}}(\mathcal{G}_n(C)))$.

Single-sorted sketches

Definition

Let $\mathcal{S} = (\mathsf{C}, \mathcal{L}, \sigma)$ be a finitary limit sketch. An object C_0 in C is called sketch-cogenerator of $\mathcal{S}$ if $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n(C_0)$. The sketch $\mathcal{S}$ is called single-sorted provided that it has a sketch-cogenerator.

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Lemma

Let $\mathcal{S} = (\mathsf{C}, \mathcal{L}, \sigma)$ be a finitary, single-sorted limit sketch with sketch-cogenerator C_0. For each object C in C, there exists a finite subset $M \subseteq \mathsf{C}(C, C_0)$ such that, for each model $F \colon \mathcal{C} \longrightarrow \mathsf{A}$ of $\mathcal{S}$, the cone $(F(f) \colon F(C) \longrightarrow F(C_0))_{f \in M}$ is a mono-cone in A.

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Corollary

Let $\mathcal{S} = (\mathsf{C}, \mathcal{L}, \sigma)$ be a finitary, single-sorted limit sketch with sketch-cogenerator C_0.

- The evaluation functor $\mathrm{ev}_{C_0}\colon \mathrm{Mod}(\mathcal{S}, \mathsf{A}) \longrightarrow \mathsf{A}$ is faithful.
- Assume that $|-|\colon \mathsf{A} \longrightarrow \mathsf{Set}$ preserves finite mono-cones and let $F\colon \mathsf{C} \longrightarrow \mathsf{A}$ be a model of $\mathcal{S}$ in A. Then $|F(C)|$ is finite for each object C in C if and only if $|F(C_0)|$ is finite.

Our starting point

Let $\mathcal{S}_A = (C_A, \mathcal{L}_A, \sigma_A)$ and $\mathcal{S}_B = (C_B, \mathcal{L}_B, \sigma_B)$ be single sorted, finitary limit sketches with sketch-cogenerators C_A and C_B.

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- The category $\mathrm{Mod}(\mathcal{S}_B, \mathrm{Set})$ is a locally finitely presentable category, hence (co)complete and (co)wellpowered and the forgetful functor $\mathrm{ev}_{C_B} \colon \mathrm{Mod}(\mathcal{S}_B, \mathrm{Set}) \longrightarrow \mathrm{Set}$ has a left adjoint and preserves filtered colimits.

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- The category $\mathrm{Mod}(\mathcal{S}_A, \mathrm{BooSp})$ is locally copresentable and therefore (co)complete and (co)wellpowered and has a cogenerating set. Hence, the functor $\mathrm{ev}_{C_A}\colon \mathrm{Mod}(\mathcal{S}_A, \mathrm{BooSp}) \longrightarrow \mathrm{BooSp}$ has a left adjoint as well.

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Furthermore, we consider objects $\widetilde{A}$ in $\mathsf{Mod}(\mathcal{S}_A, \mathsf{BooSp})$ and $\widetilde{B}$ in $\mathsf{Mod}(\mathcal{S}_2, \mathsf{Set})$ with finite underlying set $|\widetilde{A}(C_A)| = \widetilde{B}(C_B)$ are given.

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Let $\mathbb{M}_A$ and $\mathbb{M}_B$ be classes of $\mathrm{Mod}(\mathcal{S}_A, \mathrm{BooSp})$-morphisms resp. $\mathrm{Mod}(\mathcal{S}_B, \mathrm{Set})$-morphisms closed under composition, pullback and intersection stable, containing all regular monomorphisms and contained in the class of all embeddings.

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We define A as the full subcategory of $\mathrm{Mod}(\mathcal{S}_A, \mathrm{BooSp})$ of all $\mathbb{M}_A$-subobjects of powers of $\widetilde{A}$. Likewise, B denotes the full subcategory of $\mathrm{Mod}(\mathcal{S}_B, \mathrm{Set})$ of all $\mathbb{M}_B$-subobjects of powers of $\widetilde{B}$.

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Remark

A is an $\mathbb{M}_A$-ExtrEpi-reflective subcategory of $\mathrm{Mod}(\mathcal{S}_A, \mathrm{BooSp})$ with left adjoint $R_{\widetilde{A}}\colon \mathrm{Mod}(\mathcal{S}_A, \mathrm{BooSp}) \longrightarrow$ A and B is an $\mathbb{M}_B$-ExtrEpi-reflective subcategory of $\mathrm{Mod}(\mathcal{S}_B, \mathrm{Set})$ with left adjoint $R_{\widetilde{B}}\colon \mathrm{Mod}(\mathcal{S}_B, \mathrm{Set}) \longrightarrow$ B.

The "algebraic" side

Proposition

Each object B in B is a filtered colimit of finite objects in B.

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Each object B in $\mathbb{B}$ is a filtered colimit of finite objects in $\mathbb{B}$.

Proof.

- An object B in $\mathbb{B}$ is finite if and only if $\mathbb{B}(B, \widetilde{B})$ is finite.

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- Each presheaf F in $\mathrm{Set}^{\mathsf{C}_B}$ is a colimit of representables.
- For a representable presheaf $\mathsf{C}_B(C, -)$:

$$\mathsf{B}(R_{\widetilde{B}}(\mathsf{C}_B(C, -)), \widetilde{B}) = \mathrm{Nat}(\mathsf{C}_B(C, -), \widetilde{B}) = \widetilde{B}(C)$$

is finite.

□

6. Models in Boolean spaces

Copresentable objects

Remark

Since A is a reflective subcategory of $\text{Mod}(\mathcal{S}_A, \text{BooSp})$, an object A of A is finitely copresentable in A provided that it is in $\text{Mod}(\mathcal{S}_A, \text{BooSp})$.

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Lemma

Assume that C_A is finitely generated. An object M in $\mathrm{Mod}(\mathcal{S}_A, \mathrm{BooSp})$ is finitely copresentable provided that, for each C in C_A, $M(C)$ is a finite discrete space.

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The duality compactness theorem

Proposition

Let $D\colon I \longrightarrow \mathsf{A}$ be a diagram in A with limit $(p_i\colon L \longrightarrow D(i))_{i\in I}$ such that each $\eta_{D(i)}$ is an isomorphism. Then $(F(p_i)\colon F(L) \longrightarrow FD(i))_{i\in I}$ is a colimit of $FD\colon I^{\mathrm{op}} \longrightarrow \mathsf{B}$ provided that $\hom(-,\widetilde{A})$ sends $(p_i\colon L \longrightarrow D(i))_{i\in I}$ to a colimit of $\hom(D(-),\widetilde{A})\colon I^{\mathrm{op}} \longrightarrow \mathsf{Set}$.

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The "Bourbaki-criterion"

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Let $D\colon I \longrightarrow \text{CompHaus}$ be a cofiltered diagram. Then a cone $(p_i\colon L \longrightarrow D(i))_{i\in I}$ for D is a limit cone if and only if

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   That is, "the image of each p_i is as large as possible".

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Bourbaki, Nicolas (1942). Éléments de mathématique. 3. Pt. I: Les structures fondamentales de l'analyse. Livre 3: Topologie générale. Paris: Hermann & Cie.

The "Bourbaki-criterion"

Theorem

Let $D\colon I \longrightarrow \mathsf{CompHaus}$ be a cofiltered diagram. Then a cone $(p_i\colon L \longrightarrow D(i))_{i\in I}$ for D is a limit cone if and only if

1. $(p_i\colon L \longrightarrow D(i))_{i\in I}$ is mono and,
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- This characterisation applies also to BooSp.
- Recall that a cone in A is a limit cone if and only if it is initial with respect to $\mathsf{A} \longrightarrow \mathsf{BooSp}$ and it is a limit in BooSp.

The canonical diagram

For an object A in A, we consider the canonical diagram

$$\begin{aligned} D_A \colon A/\mathsf{A}_{\mathrm{fin}} &\longrightarrow \mathsf{A}. \\ (A \to A_0) &\longmapsto A_0 \end{aligned}$$

and the canonical cone $(A \longrightarrow A_0)_{A \to A_0}$.

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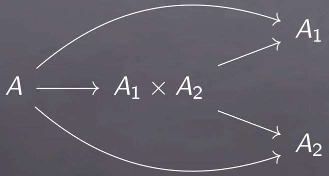

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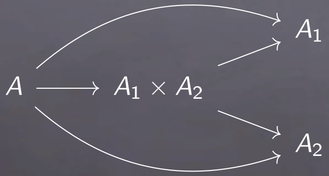
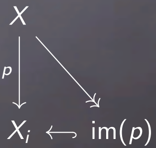


- If A has "image factorisation" then the canonical cone is a limit of the canonical diagram.

finite:



Summing up

Theorem
Our dual adjunction is a dual equivalence provided that the following hold:

- $\mathcal{C}_A$ is finitely generated and
- A has "image factorisations".

Structure switch

Example

From

$$\mathsf{BooSp} \sim \mathsf{BA}^{\mathrm{op}}$$

(induced by $(2, 2)$ we get

$$\mathsf{BooSp}_{\mathrm{fin}} \sim \mathsf{BA}^{\mathrm{op}}_{\mathrm{fin}},$$

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Well, if 2 is a cogenerator in $\mathsf{BooSpBA}$ ...

Profinite Algebras

Theorem

Consider an algebraic theory containing only "at most" binary operation symbols (finitely many) so that

- the binary operations are associative,
- there is a total order on the binary operation symbols and the distributive laws hold,
- The unitary operations are closed under composition,
- the de Morgan laws hold (for every unary and every binary operation symbol, there exist . . .).

Then every algebra in Boolean spaces is profinite.

Johnstone, Peter T. (1986). Stone spaces. Vol. 3. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press. xxii + 370. Reprint of the 1982 edition.

Part 3
Kleisli categories, Splitting idempotents, and all that

Halmos duality

Theorem

$\mathrm{BooSpKripke} \sim \mathrm{BAO}^{\mathrm{op}}$.

Boolean space Kripke frame:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ R \big\downarrow & & \big\downarrow S \\ X & \xrightarrow[f]{} & Y \end{array}$$

Jónsson, Bjarni and Tarski, Alfred (1951). "Boolean algebras with operators. I". In: American Journal of Mathematics 73.(4), pp. 891–939.

Kupke, Clemens, Kurz, Alexander, and Venema, Yde (2004). "Stone coalgebras". In: Theoretical Computer Science 327.(1-2), pp. 109–134.

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Theorem
BooSpRel $\sim \mathsf{FinSup}_{\mathsf{BA}}^{\mathrm{op}}$.

$$\begin{array}{ccc} \mathsf{BooSp} & \xrightarrow{\hom(-,2)} & \mathsf{BA}^{\mathrm{op}} \\ \big\downarrow & & \big\downarrow \\ \mathsf{BooSpRel} & \xrightarrow[\hom(-,1)]{} & \mathsf{FinSup}_{\mathsf{BA}}^{\mathrm{op}} \end{array}$$

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Halmos, Paul R. (1956). "Algebraic logic I. Monadic Boolean algebras". In: Compositio Mathematica 12, pp. 217–249.

Halmos duality (variation)

Theorem
PriestKripke $\sim$ $\mathsf{DLO}^{\mathrm{op}}$.

"Priestley Kripke frame":

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle R}\big\downarrow\!\!\!\!+ & & +\!\!\!\!\big\downarrow{\scriptstyle S} \\ X & \xrightarrow[\ f\ ]{} & Y \end{array}$$

Theorem
PriestDist $\sim$ $\mathsf{FinSup}_{\mathsf{DL}}^{\mathrm{op}}$.

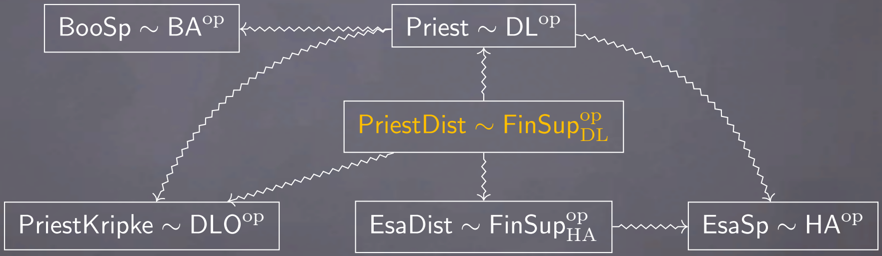
$$\begin{array}{ccc} \mathsf{Priest} & \xrightarrow{\ \hom(-,2)\ } & \mathsf{DL}^{\mathrm{op}} \\ \big\downarrow & & \big\downarrow \\ \mathsf{PriestDist} & \xrightarrow[\ \hom(-,1)\ ]{} & \mathsf{FinSup}_{\mathsf{DL}}^{\mathrm{op}} \end{array}$$

Cignoli, Roberto, Lafalce, S., and Petrovich, Alejandro (1991). "Remarks on Priestley duality for distributive lattices". In: Order 8.(3), pp. 299–315.

Petrovich, Alejandro (1996). "Distributive lattices with an operator". In: Studia Logica 56.(1-2), pp. 205–224. Special issue on Priestley duality.

Bonsangue, Marcello M., Kurz, Alexander, and Rewitzky, Ingrid M. (2007). "Coalgebraic representations of distributive lattices with operators". In: Topology and its Applications 154.(4), pp. 778–791.

The bigger picture



The powerset monad

The powerset monad $\mathbb{P} = (P, m, e)$ on Set consists of the powerset functor $P\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ and

$$e_X\colon X \longrightarrow PX, \quad x \longmapsto \{x\} \quad \text{and} \quad m_X\colon PPX \longrightarrow PX, \quad \mathcal{A} \longmapsto \bigcup \mathcal{A}.$$

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Remark

$\mathsf{Rel} \sim \mathsf{Set}_{\mathbb{P}}$.

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A relation $r\colon X \relbar\joinrel\mapsto Y$ is a function if and only if

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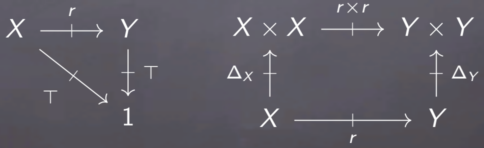
- r is a homomorphism of comonoids in the monoidal category Rel:



The Upset monad

The upset monad $\mathbb{U} = (U, m, e)$ on Ord consists of the upset functor $U\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$ defined by

$$UX = \{A \subseteq X \mid {\uparrow}A = A\}, \quad Uf\colon UX \longrightarrow UY, \quad A \longmapsto {\uparrow}f(A)$$

and

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Remark

$\mathsf{Dist} \sim \mathsf{Ord}_{\mathbb{U}}$.

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This monad restricts to BooSp and $\mathsf{BooSp}_{\mathbb{V}} \sim \mathsf{BooSpRel}$.

Vietoris monad (ordered case)

Definition

An orderered compact space is a triple $(X, \leq, \tau)$ consisting of a set X, an order $\leq$ on X and a compact Hausdorff topology τ on X so that the set

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is closed with respect to the product topology.

Nachbin, Leopoldo (1950). Topologia e Ordem. University of Chicago Press.

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More information:

Schalk, Andrea (1993). "Algebras for Generalized Power Constructions". PhD thesis. Technische Hochschule Darmstadt.

Vietoris monad (the topological case)

The lower Vietoris monad $\mathbb{V} = (V, m, e)$ on Top consists of the functor $V \colon \mathsf{Top} \longrightarrow \mathsf{Top}$ sending a topological space X to the space

$$VX = \{A \subseteq X \mid A \text{ is closed}\}$$

with the topology generated by the sets

$$B^{\Diamond} = \{A \in VX \mid A \cap B \neq \varnothing\} \quad (B \subseteq X \text{ open}),$$

and $Vf \colon VX \longrightarrow VY$ sends A to $\overline{f[A]}$, for $f \colon X \longrightarrow Y$ in Top; and the unit e and the multiplication m of $\mathbb{V}$ are given by

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Nachbin, Leopoldo (1992). "Compact unions of closed subsets are closed and compact intersections of open subsets are open". In: Portugaliæ Mathematica 49.(4), pp. 403–409.

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Remark

The classic Vietoris construction, with closed sets, does not define an obvious functor on Top. That is, adding the sets $U^{\Box}$ to the subbasis of above does not define a functor.

Stone vs. Priestley spaces

Theorem

The category Spec of spectral spaces and spectral maps is dually equivalent to the category DL of distributive lattices and homomorphisms.

$$\mathsf{Spec} \simeq \mathsf{DL}^{\mathrm{op}}.$$

Stone, Marshall Harvey (1938). "Topological representations of distributive lattices and Brouwerian logics". In: Časopis pro pěstování matematiky a fysiky 67.(1), pp. 1–25.

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Definition

A topological space X is spectral whenever X is sober and the compact and open subsets are closed under finite intersections and form a base for the topology of X.

A continuous map $f\colon X \longrightarrow Y$ between spectral spaces is called spectral whenever $f^{-1}(A)$ is compact, for every $A \subseteq Y$ compact and open.

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In particular: $\mathsf{Spec} \sim \mathsf{Priest}$

Stably compact spaces

Definition

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Remark

Every compact Hausdorff space is stably compact and every continuous map between compact Hausdorff spaces is spectral:

$$\mathsf{CompHaus} \longrightarrow \mathsf{StablyComp}.$$

Connection with ordered compact spaces

Remark

This functor has a right adjoint

$$\mathsf{StablyComp} \longrightarrow \mathsf{CompHaus}$$

which sends a stably compact space X to the compact Hausdorff space with the same underlying set and the patch topology: the topology generated by the open subsets and the complements of the compact down-closed subsets of X.

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Theorem

Every stably compact space X defines an ordered compact Hausdorff space with the patch topology and the underlying order of X, and an ordered compact Hausdorff space X becomes a stably compact space where the topology is given by all down-closed opens of X.

$$\mathsf{PosComp} \sim \mathsf{StablyComp}$$

Back to Vietoris

Proposition

- The monad $\mathbb{V} = (V, m, e)$ on Top is of Kock-Zöberlein type, that is, $e_{VX} \leq Ve_X$ or, equivalently, $e_{VX} \dashv m_X \dashv Ve_X$.

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Corollary

Consequently, the monad $\mathbb{V} = (V, m, e)$ on Top restricts to monads on StablyComp and on Spec.

Back to Vietoris

Remark

Using the adjunction between StablyComp and CompHaus, we can transfer the monad $\mathbb{V}$ on StablyComp to the Vietoris monad $\mathbb{V}$ on CompHaus.

The topology of VX is the patch topology which is generated by the sets

$$U^{\Diamond} = \{A \subseteq X \mid A \cap B = \varnothing\} \quad (U \subseteq X \text{ open}) \quad \text{and}$$
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A compact Hausdorff space X is a Stone space if and only if VX is a Stone space.

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Therefore the monad $\mathbb{V}$ on CompHaus restricts to a monad on BooSp.

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- Identify $\mathbb{D}$, that is, find a "nice" monad isomorphic to $\mathbb{D}$.

Table of content

7. Halmos dualities

Liftings to Kleisli categories

Theorem

Let X and A be categories with respresentable forgetful functors to Set, $\mathbb{T} = (\mathsf{T}, m, e)$ a monad on X and $F \dashv G$ an adjunction

$$\mathsf{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{A}^{\mathrm{op}} \quad (\bot)$$

. induced by $(\widetilde{X}, \widetilde{A})$. The following data are in bijection.

(i) Functors $F\colon \mathsf{X}_{\mathbb{T}} \longrightarrow \mathsf{A}^{\mathrm{op}}$ commuting with the left adjoints.

(ii) Monad morphisms $j\colon \mathbb{T} \longrightarrow \mathbb{D}$ ($\mathbb{D}$ induced by $F \dashv G$).

(iii) $\mathbb{T}$-algebra structures $\sigma\colon T\widetilde{X} \longrightarrow \widetilde{X}$ such that the map

$$\widehat{(-)}\colon \mathsf{X}(X, \widetilde{X}) \longrightarrow \mathsf{X}(TX, \widetilde{X}), \quad \psi \longmapsto \sigma \cdot T\psi =: \widehat{\psi}$$

is an A-morphism $\kappa_X\colon FX \longrightarrow FTX$.

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Remark

For every X in X:

$$\begin{array}{ccc} |TX| & \xrightarrow{j_X} & \mathsf{A}(FX,\widetilde{A}) \\ & \searrow^{\widehat{\psi}} & \downarrow \mathrm{ev}_{FX,\psi} \\ & & |\widetilde{X}| = |\widetilde{A}| \end{array}$$

Hence, j_X is an embedding if and only if the cone

$$(\widehat{\psi}\colon TX \longrightarrow \widetilde{X})_\psi$$

is point-separating and initial.

Some simplification

If $\widetilde{X} = TX_0$ with $\mathbb{T}$-algebra structure m_{X_0}, then

- the functor $F\colon \mathsf{X}_{\mathbb{T}} \longrightarrow \mathsf{A}^{\mathrm{op}}$ is a lifting of the hom-functor $\mathsf{X}(-, X_0)\colon \mathsf{X}_{\mathbb{T}} \longrightarrow \mathsf{Set}^{\mathrm{op}}$,

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- interpreting the elements of TX as morphisms $\varphi : X_0 \longrightarrow\!\!\!\!\!\!\!+\;\; X$ in the Kleisli category $\mathsf{X}_{\mathbb{T}}$ allows to describe the components of the monad morphism j using composition in $\mathsf{X}_{\mathbb{T}}$:

$$j_X \colon |TX| \longrightarrow \mathrm{hom}(FX, \widetilde{A}), \quad \varphi \longmapsto (\psi \mapsto \psi \cdot \varphi).$$

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The monad morphism j is given by

$$j_X : VX \longrightarrow \mathsf{SFrm}_{\bigvee}(FX, 2), \quad A \longmapsto (B \mapsto [\![A \cap B \neq \varnothing]\!])$$

hence j is an isomorphism and we obtain $\mathsf{Top}_{\mathbb{V}} \simeq \mathsf{SFrm}_{\bigvee}^{\mathrm{op}}$.

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Compactness guarantees that j_X is surjective, hence

$$\mathsf{Spec}_{\mathbb{V}} \sim \mathsf{FinSup}_{\mathrm{DL}}^{\mathrm{op}}.$$

8. Idempotent split completion

Splitting idempotents

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Example
The category Rel is not idempotent split complete.

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Lemma

Let A be a full subcategory of B and assume that idempotents split in B. Let $\overline{\mathsf{A}}$ be the full subcategory of B defined by the retracts of the objects in A. Then idempotents split in $\overline{\mathsf{A}}$ and $\mathsf{A} \to \overline{\mathsf{A}}$ is the free idempotent split completion of A.

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Remark

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Theorem
For a morphism $f: X \longrightarrow Y$ in StablyComp, the following assertions are equivalent.

(i) f is down-wards open.

(ii) The spectral distributor $f_*: X \relbar\!\!\!\circ\!\!\!\rightarrow Y$ has a right adjoint in StablyCompDist.

(iii) the distributor $f^*: Y \relbar\!\!\!\circ\!\!\!\rightarrow X$ is a spectral distributor.

Esakia spaces

Remark

The Priestley spaces corresponding to Heyting algebras are the Esakia spaces: those Priestley spaces X where the down-closure of every open subset of X is again open.

Esakia, Leo (1974). "Topological Kripke models". In: Doklady Akademii Nauk SSSR 214, pp. 298–301.

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Definition

A stably compact space X is called an Esakia space whenever, for every open subset A of the patch space X_p of X, its down-closure $\downarrow A$ is open in X.

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We write GEsaDist to denote the full subcategory of StablyCompDist defined by all Esakia spaces, and EsaDist stands for the full subcategory of GEsaDist defined by all spectral spaces.

Esakia spaces split Boolean spaces

Theorem
For a stably compact space X, the following assertions are equivalent.

(i) X is an Esakia space.

(ii) The spectral map $i\colon X_p \longrightarrow X, \quad x \longmapsto x$ is down-wards open.

(iii) The spectral distributor $i_*\colon X_p \multimap X$ has a right adjoint (necessarily given by i^*).

(iv) X is a split subobject of a compact Hausdorff space Y in StablyCompDist.

If X is spectral, then the space Y in the last assertion can be chosen as a Stone space.

Easkia spaces are idempotent split complete

Remark
Recall that $\mathrm{SpecDist} \simeq \mathrm{FinSup}_{\mathrm{DL}}^{\mathrm{op}}$. Moreover, the category $\mathrm{FinSup}_{\mathrm{DL}}$ is idempotent split complete

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Remark

Recall that SpecDist $\simeq \mathrm{FinSup}_{\mathrm{DL}}^{\mathrm{op}}$. Moreover, the category $\mathrm{FinSup}_{\mathrm{DL}}$ is idempotent split complete, and therefore SpecDist is idempotent split complete.

Corollary

The category EsaDist is the idempotent split completion of BooSpRel.

co-Heyting algebras

Remark

For a distributive lattice L, we consider its Booleanisation $j\colon L \longrightarrow B$ which is given by any epimorphic embedding in DL of L into a Boolean algebra B.

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For a distributive lattice L, the following assertions are equivalent.

1. L is a co-Heyting algebra.
2. The lattice homomorphism $j\colon L \longrightarrow B$ has a left adjoint in $\mathrm{FinSup}_{\mathrm{DL}}$ $j^{+}\colon B \longrightarrow L$.
3. L is a split subobject of a Boolean algebra in $\mathrm{FinSup}_{\mathrm{DL}}$.

McKinsey, John C. C. and Tarski, Alfred (1946). "On closed elements in closure algebras". In: Annals of Mathematics. Second Series 47.(1), pp. 122–162.

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The category $\mathrm{FinSup}_{\mathrm{coHeyt}}$ is the idempotent split completion of $\mathrm{FinSup}_{\mathrm{BA}}$.

Esakia duality

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The equivalence $\mathsf{SpecDist} \sim \mathsf{FinSup}_{\mathrm{DL}}^{\mathrm{op}}$ restricts to an equivalence

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A lattice homomorphism $f\colon L_1 \longrightarrow L_2$ between co-Heyting algebras preserves the co-Heyting operation if and only if

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A lattice homomorphism $f\colon L_1 \longrightarrow L_2$ between co-Heyting algebras preserves the co-Heyting operation if and only if the diagram

$$\begin{array}{ccc} B_1 & \xrightarrow{\overline{f}} & B_2 \\ {\scriptstyle j_1^+}\downarrow & & \downarrow{\scriptstyle j_2^+} \\ L_1 & \xrightarrow[f]{} & L_2 \end{array}$$

commutes.

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Remark

A lattice homomorphism $f\colon L_1 \longrightarrow L_2$ between co-Heyting algebras preserves the co-Heyting operation if and only if the corresponding spectral map $g\colon X_1 \longrightarrow X_2$ makes the diagram of spectral distributors

$$\begin{array}{ccc} X_1 & \xrightarrow{\;g_*\;} & X_2 \\ {\scriptstyle i_1^*}\downarrow & & \downarrow{\scriptstyle i_2^*} \\ (X_1)_p & \xrightarrow[\;g\;]{} & (X_2)_p \end{array}$$

commutative. Element-wise: for all $x \in X_1$ and $y \in X_2$ with $g(x) \leq y$, there is some $x' \in X_1$ with $x \leq x'$ and $g(x') = y$.

One more ...

Rosebrugh, Robert and Wood, Richard J. (1994). "Constructive complete distributivity IV". In: Applied Categorical Structures 2.(2), pp. 119–144.

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