Algebraizable Weak Logics

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In this talk we shall see:

- The notion of algebraizability and its scope;
- A generalization of the standard framework to weak logics;
- Application of our framework to inquisitive and dependence logic.

Logic as a consequence relation

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A logic of type $\mathcal{L}$ is a consequence relation $\vdash$ on the set $\mathcal{F}m_{\mathcal{L}}$ that is closed under uniform substitution:

3. For all substitutions σ, if $\Gamma \vdash \phi$, then $\sigma[\Gamma] \vdash \sigma[\phi]$.

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Let $\tau : \mathcal{F}m \to \mathcal{P}(\mathrm{Eq})$ and $\Delta : \mathrm{Eq} \to \mathcal{P}(\mathcal{F}m)$ be functions commuting with substitutions of Eq and $\mathcal{F}m$. We call them structural transformers.

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$$\Gamma \vdash \phi \iff \tau[\Gamma] \vDash_{\mathbf{Q}} \tau(\phi) \quad \text{(Alg1)}$$

$$\Delta[\Theta] \vdash \Delta(\eta, \delta) \iff \Theta \vDash_{\mathbf{Q}} \eta \approx \delta \quad \text{(Alg2)}$$

$$\phi \dashv\vdash \Delta[\tau(\phi)] \quad \text{(Alg3)}$$

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We then call $\mathbf{Q}$ the equivalent algebraic semantics for $\vdash$.

Theorem (Uniqueness)

If the tuples $(\mathbf{Q}_0, \tau_0, \Delta_0)$ *and* $(\mathbf{Q}_1, \tau_1, \Delta_1)$ *both witness the algebraizability of a standard logic* $\vdash$*, then:*

1. $\mathbf{Q}_0 = \mathbf{Q}_1$;
2. $\Delta_0(x, y) \dashv\vdash \Delta_1(x, y)$;
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- $\mathtt{CPC}$ is algebraized by **BA**, $\tau(x) := \{x \approx 1\}$, $\Delta(x, y) = \{x \to y, y \to x\}$.
- $\mathtt{IPC}$ is algebraized by **HA**, $\tau(x) := \{x \approx 1\}$, $\Delta(x, y) = \{x \to y, y \to x\}$.
- $\mathtt{K}_l$ is not algebraizable,

$$\mathtt{K}_l = \{(\Gamma, \phi) : \forall \langle W, R, v\rangle, \forall\, w \in W,\ \text{if } w \Vdash \Gamma \text{ then } w \Vdash \phi\}.$$

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Interestingly, many of these logics have been investigated from an algebraic perspective. Can we then treat them with the tools of abstract algebraic logic?

Weak Logics and Expanded Algebras

Algebraizability of Weak Logics

Applications to Inquisitive (Dependence) Logic

Weak Logics

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This generalises the notion of weak logic from Ciardelli 2009.

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If $\mathbf{Q}$ is a class of expanded algebras and $\Theta \cup \{\epsilon \approx \delta\}$ a set of equations, we define:

$$\begin{aligned}\Theta \vDash^{c}_{\mathbf{Q}} \epsilon \approx \delta \iff & \text{ for all } \mathcal{A} \in \mathbf{Q}, \\ & \text{ for all } h \in \mathtt{Hom}(\mathcal{F}m, \mathcal{A}), \text{ s.t. } h[\mathtt{Var}] \subseteq \text{core}(\mathcal{A}) \\ & \text{ if } h(\epsilon_i) = h(\delta_i) \text{ for all } \epsilon_i \approx \delta_i \in \Theta, \text{ then } h(\epsilon) = h(\delta).\end{aligned}$$

For any set of equations $\Sigma = \{\epsilon_i(x) \approx \delta_i(x) : i \leq n\}$ we let:

$$\Sigma(x, \mathcal{A}) := \{x \in \mathcal{A} : \mathcal{A} \vDash \epsilon_i(x) \approx \delta_i(x) \text{ for all } i \leq n\}.$$

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A class of expanded algebras $\mathbf{K}$ is (uniformly) equationally definable if there is some finite set of equations Σ such that for all $\mathcal{A} \in \mathbf{K}$, $\text{core}(\mathcal{A}) = \Sigma(x, \mathcal{A})$.

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3. *The induced consequence relation $\vDash^c_{\mathbf{Q}}$ is compact.*

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Theorem (Maltsev Theorem for Core-Generated Quasivarieties)

Let $\mathbf{Q}$ be a quasi-variety of expanded algebras and let $\mathcal{A}$ be core-generated, then:

$$\mathcal{A} \in \mathbf{Q}_{CG} \iff \mathcal{A} \models^c Th^c(\mathbf{Q}).$$

Weak Logics and Expanded Algebras

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A weak logic $\vdash$ is algebraizable if there are structural transformers $\tau : \mathcal{F}m \to \wp(\mathrm{Eq})$ and $\Delta : \mathrm{Eq} \to \wp(\mathcal{F}m)$ and a core-generated, equationally defined quasivariety $\mathbf{Q}$ such that:

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$$\Gamma \vdash \phi \iff \tau[\Gamma] \vDash^{c}_{\mathbf{Q}} \tau(\phi) \qquad \text{(Weak-Alg1)}$$

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We then say that **Q** is the equivalent algebraic semantics of $\vdash$.

Theorem (Uniqueness of Equivalent Semantics)

If $(\mathbf{Q}_0, \tau_0, \Delta_0, \Sigma_0)$ *and* $(\mathbf{Q}_1, \tau_1, \Delta_1, \Sigma_1)$ *witness the algebraizability of a weak logic* $\vdash$, *then for* $i \in \{0, 1\}$*:*

(1) $\mathbf{Q}_0 = \mathbf{Q}_1$

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Proof.

(sketch) Using the previous version of Maltsev's theorem. □

Characterization of Algebraizability (i)

Let $\vdash$ be a weak logic, we define its schematic fragment as follows:

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Proof (sketch).

Characterization of Algebraizability (i)

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We say that a weak logic $\vdash$ is finitely representable if there is a finite set of formulae Λ such that for all $\Gamma \cup \{\phi\} \subseteq \mathcal{F}m$:

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- ($\Leftarrow$) Verify that $(\mathbf{Q}, \tau, \Delta, \tau(\Lambda))$ algebraizes $\vdash$. □

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Theorem (Blok, Pigozzi)

Let $\vdash$ be a standard logic and $\mathbf{Q}$ a quasi-variety, then the following are equivalent:

1. $\vdash$ *is algebraizable with equivalent algebraic semantics* $\mathbf{Q}$*;*
2. $Fi_{\vdash}(\mathcal{A}) \cong Con_{\mathbf{Q}}(\mathcal{A})$*, for any algebra* $\mathcal{A}$*;*
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Theorem (Isomorphism Theorem for Weak Logics)

Let $\vdash$ be a weak logic and $\mathbf{Q}$ a core-generated quasi-variety of expanded algebras with core defined by Σ. The following are equivalent:

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3. $Th(\mathrm{Schm}(\vdash)) \cong Th(\vDash_{\mathbf{Q}})$ *and there are finite* $\Lambda \subseteq \mathcal{F}m$*,* $\Sigma \subseteq \mathrm{Eq}$ *s.t.* $Th^{\Lambda}(\mathrm{Schm}(\vdash)) = Th(\vdash)$ *and* $Th(\vDash^{c}_{\mathbf{Q}}) = Th^{\Sigma}(\vDash_{\mathbf{Q}})$*.*

Weak Logics and Expanded Algebras

Algebraizability of Weak Logics

Applications to Inquisitive (Dependence) Logic

Inquisitive logics InqB and InqI

Inquisitive (dependence) logics are typically defined via *team semantics* – formulas are evaluated with respect to set of assignments.

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Then InqI is axiomatized by:

$$\texttt{InqI} := \textit{(an axiomatization of)}\ \texttt{IPC} + (\alpha \to (\phi \vee \psi)) \to ((\alpha \to \phi) \vee (\alpha \to \psi))$$

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We can obtain the classical variant InqB as InqI $+ \{\neg\neg\alpha \to \alpha : \alpha \text{ is standard}\}$, closing under MP.

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We can obtain the classical variant $\texttt{InqB}$ as $\texttt{InqI} + \{\neg\neg\alpha \to \alpha : \alpha \text{ is standard}\}$, closing under $\texttt{MP}$.

Fact

$\texttt{InqB}$ *and* $\texttt{InqI}$ *are not closed under uniform substitution.*

Algebraizability of InqB

We recall the following facts from the literature (Ciardelli 2009; Bezhanishvili, Grilletti, and Holliday 2019; Bezhanishvili, Grilletti, and Quadrellaro 2021):

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$\mathtt{InqB}$ *is algebraizable.*

Proof.

It suffices to consider the following witnesses:

- $\mathit{Var}(\mathtt{ML})$;
- $\Sigma := \{x \approx \neg\neg x\}$;
- $\tau(\phi) = \phi \approx 1$;
- $\Delta(x, y) = x \leftrightarrow y$.

□

Non-Algebraizability of `InqI`

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InqI *is not algebraizable.*

Proof.

- (sketch) Suppose InqI is algebraized by $(\mathbf{Q}, \tau, \Delta, \Sigma)$. The standard logic of $\mathbf{Q}$, Schm(InqI), is an intermediate logic and algebraized by $(\mathbf{Q}, \phi \approx 1, x \leftrightarrow y)$, for $\mathbf{Q}$ a subvariety of Heyting algebras.

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- Over InqI, we have wlog $\Sigma \subseteq \{\bigvee_{i \leq n} \tau_i(x) \approx 1 : \tau_i \text{ is } \vee\text{-free}\}$. By DP $\Sigma \subseteq \{\tau_i(x) \approx 1 : \tau_i \text{ is } \vee\text{-free}\}$.
- For any suitable candidate $\tau_i(x)$, we show by the usual semantics of InqI that InqI $\nvdash \tau_i(x)$, contradiction. □

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- Extension of our setting to non-algebraizable weak logics, e.g `InqI`.
- Applications to other logics without uniform substitution.

Thank you for your attention!

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