Łukasiewicz logic properly displayed

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General plan

Mathematical fuzzy logics: reasoning about truth degrees.

Hilbert systems: tool for presenting logics corresponding to algebras.

Structural proof theory studies structure and properties of proofs.

Sequent calculi: tool for organizing proofs as to preserve *analitycity*.
Recent line of research: *algorithmic generation of rules*.

Problem: The distinctive axiom of Łukasiewicz logic is NOT analytic-inductive (not even canonical).

Desiderata (work in progress): refinement of the general theory where

- √ logical rules reflect basic order-theoretic properties
  - ⇝ division of labour
- √ the specific features of the logic are captured by structural rules
  - ⇝ modularity
- √ all rules are automatically generated via the algorithm ALBA
  - ⇝ uniformity
- × canonical cut elimination ???

General methodology

Multi-type (algebraic) proof theory

- canonical extensions — algebra
- unified correspondence theory — duality
- proper display calculi — structural proof theory

Proof calculi with a uniform metatheory

- (supporting an **inferential theory of meaning**)
- canonical **cut elimination** and **subformula property**
- **soundness**, **completeness**, **conservativity**

Range

- LEs and their analytic-inductive axiomatic extensions
- if not analytic-inductive, provide a multi-type presentation

Examples: bi-lattices, semi-De Morgan logic, (monotone) modal logics, dynamic epistemic logic, linear logic, non classical (first order) logics. . .

- Łukasiewicz logic ???

Łukasiewicz connectives

(standard) evaluations are $v : \mathrm{Form} \longrightarrow [0,1]$.

$$
\begin{array}{rcccl}
(A \to B) \to B & \equiv & A \vee B & = & \max\{a, b\} \\
A \odot (A \to B) \equiv A \ominus (A \ominus B) & \equiv & A \wedge B & = & \min\{a, b\} \\
& & & & \\
\neg A \to B & \equiv & A \oplus B & = & \min\{1, a + b\} \\
\neg(A \to \neg B) \equiv \neg(\neg A \oplus \neg B) & \equiv & A \odot B & = & \max\{0, a + b - 1\} \\
\neg A \oplus B & \equiv & A \to B & = & \min\{1, 1 - a + b\} \\
\neg(A \to B) & \equiv & A \ominus B & = & \max\{0, a - b\} \\
& & & & \\
A \to \mathbf{0} & \equiv & \neg A & = & 1 - a \\
\mathbf{1} & \equiv & \neg \mathbf{0} & = & 1
\end{array}
$$

Algebraic semantics and logic

An MV-algebra $\langle X, \oplus, \neg, 0\rangle$ is a set X s.t.:

MV1 $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

MV2 $x \oplus y = y \oplus x$

MV3 $x \oplus 0 = x$

MV4 $\neg\neg x = x$

MV5 $x \oplus \neg 0 = \neg 0$

MV6 $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

Examples: (i) $[0,1]$ with $\oplus$ and $\neg$ as defined above. Subalgebras of $[0,1]$: (ii) The fragment $\langle X, \vee, \neg, \bot\rangle$ of a Boolean algebra, (iii) The rational numbers in $[0,1]$, and (iv) n-el. set $\{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$.

A Hilbert-style axiomatization in the fragment $\{\rightarrow, \mathbf{0}\}$ is the following:

Ł1 $A \rightarrow (B \rightarrow A)$

Ł2 $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

Ł3 $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$ recall $(B \rightarrow A) \rightarrow A \equiv B \vee A$

Ł4 $((A \rightarrow \mathbf{0}) \rightarrow (B \rightarrow \mathbf{0})) \rightarrow (B \rightarrow A)$

Display sequent calculi

- Natural generalization of Gentzen's sequent calculi;
- sequents $X \Rightarrow Y$, where X and Y are **structures**:
  - formulas are **atomic structures**
  - built-up: **structural connectives** (generalizing comma in sequents $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$)
  - generation **trees** (generalizing sets, multisets, sequences)
- **Display property**:

$$\dfrac{\dfrac{\dfrac{Y \Rightarrow X > Z}{X \,;\, Y \Rightarrow Z}}{Y \,;\, X \Rightarrow Z}}{X \Rightarrow Y > Z}$$

display rules semantically justified by **adjunction** / **residuation** / **Galois connection**

- **Canonical proof of cut elimination (via metatheorem)**

Multi-type proper display calculi

Definition

A **proper display calculus** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** structure-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation)
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters within each type (Properness!);
5. **reduction strategy** exists when cut formulas are principal.
6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

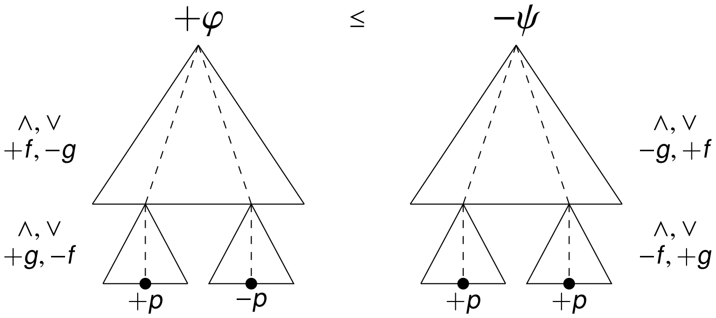
Theorem (Canonical!)

Cut elimination and subformula property hold for any **proper display calculus**.

Which logics are properly displayable?

Complete characterization:

1. the logics of any **basic** normal (D)LE;
2. axiomatic extensions of these with **analytic inductive inequalities**:
   ⇝ unified correspondence



Analytic inductive ⇒ Inductive ⇒ Canonical

Fact: cut-elim., subfm. prop., sound-&-completeness, conservativity **guaranteed** by metatheoem + ALBA-technology.

Examples

The definition of **analytic inductive inequalities** is uniform in each signature.

- Analytic inductive axioms

  $(A \to B) \vee (B \to A)$

  $(\Diamond A \to \Box B) \to \Box(A \to B)$

- Sahlqvist but non-analytic axioms

  $A \to \Diamond\Box A$

  $(\Box A \to \Diamond B) \to (A \to B)$

Łukasiewicz operators: basic properties

normal binary diamond	normal binary box
$A \odot \mathbf{0} = \mathbf{0} = \mathbf{0} \odot A$ $(A \vee B) \odot C = (A \odot C) \vee (B \odot C)$ $C \odot (A \vee B) = (C \odot A) \vee (C \odot B)$	$A \oplus \mathbf{1} = \mathbf{1} = \mathbf{1} \oplus A$ $(A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C)$ $C \oplus (A \wedge B) = (C \oplus A) \wedge (C \oplus B)$
$A \ominus \mathbf{1} = \mathbf{0} = \mathbf{0} \ominus A$ $(A \vee B) \ominus C = (A \ominus C) \vee (B \ominus C)$ $C \ominus (A \wedge B) = (C \ominus A) \vee (C \ominus B)$	$A \to \mathbf{1} = \mathbf{1} = \mathbf{0} \to A$ $(A \vee B) \to C = (A \to C) \wedge (B \to C)$ $C \to (A \wedge B) = (C \to A) \wedge (C \to B)$

residuation

$$A \odot B \leq C \quad \text{iff} \quad B \leq A \to C$$
$$C \leq B \oplus A \quad \text{iff} \quad C \ominus A \leq B$$

The basic language of D.Ł

Our choice:

- a fully residuated structural language
- non-positional reading of structural symbols

$$
\begin{array}{rrcl}
\text{formulas} & A & ::= & p \\
& & & A \wedge A \mid A \vee A \\
& & & \mathbf{1} \mid \mathbf{0} \\
& & & A \odot A \mid A \oplus A \mid A \rightarrow A \mid A \ominus A \mid \neg A \\
\text{structures} & X & ::= & A \\
& & & X \hat{\wedge} X \mid X \check{\vee} X \mid X \check{\supset} X \mid X \hat{\subset} X \\
& & & \hat{\mathbf{1}} \mid \check{\mathbf{0}} \\
& & & X \hat{\odot} X \mid X \check{\oplus} X \mid X \check{\rightarrow} X \mid X \hat{\ominus} X \mid \tilde{\neg} X
\end{array}
$$

	additive				multiplicative						
structural connectives	$\hat{\wedge}$	$\check{\vee}$	$\check{\supset}$	$\hat{\subset}$	$\hat{\mathbf{1}}$	$\check{\mathbf{0}}$	$\hat{\odot}$	$\check{\oplus}$	$\check{\rightarrow}$	$\hat{\ominus}$	$\tilde{\neg}$
logical connectives	$\wedge$	$\vee$	$(\supset)$	$(\subset)$	$\mathbf{1}$	$\mathbf{0}$	$\odot$	$\oplus$	$\rightarrow$	$\ominus$	$\neg$

$\Rightarrow$ is a preorder

- Identity and Cut rules (preorder)

$$\text{Id}\ \frac{}{p \Rightarrow p} \qquad \frac{X \Rightarrow A \qquad A \Rightarrow Y}{X \Rightarrow Y}\ \text{Cut}$$

Display postulates

adjunction / residuation / Galois connection

$$\begin{array}{c} X \mathbin{\hat{\odot}} Y \Rightarrow Z \\ \hline\hline Y \Rightarrow X \mathbin{\check{\rightarrow}} Z \end{array} \qquad \begin{array}{c} Z \Rightarrow Y \mathbin{\check{\oplus}} X \\ \hline\hline Z \mathbin{\hat{\ominus}} X \Rightarrow Y \end{array}$$

$$\begin{array}{c} \tilde{\neg} X \Rightarrow Y \\ \hline\hline \tilde{\neg} Y \Rightarrow X \end{array} \qquad \begin{array}{c} X \Rightarrow \tilde{\neg} Y \\ \hline\hline Y \Rightarrow \tilde{\neg} X \end{array}$$

Logical rules

arity and tonicity

$$\frac{A \hat{\odot} B \Rightarrow X}{A \odot B \Rightarrow X} \qquad \frac{X \Rightarrow A \quad Y \Rightarrow B}{X \hat{\odot} Y \Rightarrow A \odot B}$$

$$\frac{A \Rightarrow X \quad B \Rightarrow Y}{A \oplus B \Rightarrow X \check{\oplus} Y} \qquad \frac{X \Rightarrow A \check{\oplus} B}{X \Rightarrow A \oplus B}$$

$$\frac{X \Rightarrow A \quad B \Rightarrow Y}{A \rightarrow B \Rightarrow X \check{\rightarrow} Y} \qquad \frac{X \Rightarrow A \check{\rightarrow} B}{X \Rightarrow A \rightarrow B}$$

$$\frac{B \Rightarrow Y \quad X \Rightarrow A}{Y \hat{\ominus} X \Rightarrow B \ominus A} \qquad \frac{B \hat{\ominus} A \Rightarrow X}{B \ominus A \Rightarrow X}$$

$$\frac{\tilde{\neg} A \Rightarrow X}{\neg A \Rightarrow X} \qquad \frac{X \Rightarrow \tilde{\neg} A}{X \Rightarrow \neg A}$$

$$\frac{}{\mathbf{0} \Rightarrow \check{\mathbf{0}}} \quad \frac{X \Rightarrow \check{\mathbf{0}}}{X \Rightarrow \mathbf{0}} \qquad \frac{\hat{\mathbf{1}} \Rightarrow X}{\mathbf{1} \Rightarrow X} \quad \frac{}{\hat{\mathbf{1}} \Rightarrow \mathbf{1}}$$

Structural rules

logic specific

$$\text{w}\ \frac{X \Rightarrow Y}{X \mathbin{\hat{\odot}} Z \Rightarrow Y} \qquad \frac{X \Rightarrow Y}{X \Rightarrow Y \mathbin{\check{\oplus}} Z}\ \text{w} \qquad \text{e}\ \frac{X \mathbin{\hat{\odot}} Y \Rightarrow Z}{Y \mathbin{\hat{\odot}} X \Rightarrow Z} \qquad \frac{Z \Rightarrow X \mathbin{\check{\oplus}} Y}{Z \Rightarrow Y \mathbin{\check{\oplus}} X}\ \text{e}$$

$$\text{a}\ \frac{(X \mathbin{\hat{\odot}} Y) \mathbin{\hat{\odot}} Z \Rightarrow W}{X \mathbin{\hat{\odot}} (Y \mathbin{\hat{\odot}} Z) \Rightarrow W} \qquad \frac{W \Rightarrow (X \mathbin{\check{\oplus}} Y) \mathbin{\check{\oplus}} Z}{W \Rightarrow X \mathbin{\check{\oplus}} (Y \mathbin{\check{\oplus}} Z)}\ \text{a}$$

$$\tilde{\neg}\tilde{\neg}\ \frac{X \Rightarrow Y}{\tilde{\neg}\tilde{\neg} X \Rightarrow Y} \qquad \frac{X \Rightarrow Y}{\tilde{\neg} Y \Rightarrow \tilde{\neg} X}\ \mathit{cont} \qquad \frac{X \Rightarrow Y}{X \Rightarrow \tilde{\neg}\tilde{\neg} Y}\ \tilde{\neg}\tilde{\neg}$$

$$\tilde{\neg}\ \frac{Z \Rightarrow X \mathbin{\check{\oplus}} Y}{\tilde{\neg} X \mathbin{\hat{\odot}} Z \Rightarrow Y} \qquad \frac{X \mathbin{\hat{\odot}} Y \Rightarrow Z}{Y \Rightarrow \check{\neg} X \mathbin{\check{\oplus}} Z}\ \tilde{\neg}$$

$$\frac{X \Rightarrow Z \qquad W \Rightarrow Y}{\hat{\mathbf{1}} \Rightarrow (X \mathbin{\check{\rightarrow}} Y) \mathbin{\check{\vee}} (W \mathbin{\check{\rightarrow}} Z)}\ \text{pre}$$

Uniformity and modularity

Łukasiewicz can be presented as the (exponential-free fragment of) affine linear logic expanded with $\ominus$ + lattice distributivity + prelinearity + Ł3.

This presentation charts Łukasiewicz logic as sub-structural logic in a modular way, where all axioms are analytic-inductive but Ł3.

Prelinearity is derivable using the ALBA-generated structural rule *pre*:

$$
\begin{array}{c}
A \Rightarrow A \qquad B \Rightarrow B \\
\hline
\hat{\mathbf{1}} \Rightarrow (A \check{\rightarrow} B) \,\check{\vee}\, (A \check{\rightarrow} B) \quad \text{pre} \\
\hline
\mathbf{1} \Rightarrow (A \check{\rightarrow} B) \,\check{\vee}\, (A \check{\rightarrow} B) \\
\hline
\mathbf{1} \,\hat{\ominus}\, (A \check{\rightarrow} B) \Rightarrow A \check{\rightarrow} B \\
\hline
\mathbf{1} \,\hat{\ominus}\, (A \check{\rightarrow} B) \Rightarrow A \rightarrow B \\
\hline
\mathbf{1} \Rightarrow (A \rightarrow B) \,\check{\vee}\, (A \check{\rightarrow} B) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathbf{1} \Rightarrow (A \rightarrow B) \,\check{\vee}\, (A \rightarrow B) \\
\hline
\mathbf{1} \Rightarrow (A \rightarrow B) \vee (A \rightarrow B)
\end{array}
$$

Some variations on negation

$$\frac{\hat{\neg}_f A \Rightarrow X}{\neg_f A \Rightarrow X} \qquad \frac{A \Rightarrow X}{\hat{\neg}_f X \Rightarrow \neg_f A} \qquad \frac{X \Rightarrow A}{\neg_g A \Rightarrow \check{\neg}_g X} \qquad \frac{X \Rightarrow \check{\neg}_g A}{X \Rightarrow \neg_g A}$$

pseudo cont
$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\hat{\neg}_f A \Rightarrow \check{\neg}_g A}}{\neg_f A \Rightarrow \check{\neg}_g A}}{\neg_f A \Rightarrow \neg_g A}$$

pseudo double neg
$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\neg_g A \Rightarrow \check{\neg}_g A}}{A \Rightarrow \check{\neg}_g \neg_g A}}{\hat{\neg}_f \check{\neg}_g \neg_g A \Rightarrow \neg_f A}}{\neg_g A \Rightarrow \neg_f A}$$

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\hat{\neg} A \Rightarrow \check{\neg} A}}{\neg A \Rightarrow \check{\neg} A}}{\neg A \Rightarrow \neg A} \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A}{\tilde{\neg} A \Rightarrow \tilde{\neg} A}}{\neg A \Rightarrow \tilde{\neg} A}}{\neg A \Rightarrow \neg A}$$

Łukasiewicz operators: additional properties

regular binary diamond	regular binary box
$(A \vee B) \oplus C = (A \oplus C) \vee (B \oplus C)$ $C \oplus (A \vee B) = (C \oplus A) \vee (C \oplus B)$	$(A \wedge B) \odot C = (A \odot C) \wedge (B \odot C)$ $C \odot (A \wedge B) = (C \odot A) \wedge (C \odot B)$
$(A \wedge B) \to C = (A \to C) \vee (B \to C)$ $C \to (A \vee B) = (C \to A) \vee (C \to B)$	$(A \wedge B) \ominus C = (A \ominus C) \wedge (B \ominus C)$ $C \ominus (A \vee B) = (C \ominus A) \wedge (C \ominus B)$

The full language and a first attempt to define D.Ł

We expand the language of D.Ł with the following structural symbols:

$$\check{\odot}, \quad \hat{\oplus}, \quad \check{\ominus}, \quad \hat{\rightarrow}.$$

We extend D.Ł with the following rules:

- Display Postulates
  (notice that $\hat{\oplus}$ can be introduced only via the rule Ł3: see next slide)

$$\frac{X \mathbin{\hat{\oplus}} Y \Rightarrow Z}{X \Rightarrow Z \mathbin{\check{\ominus}} Y} \qquad \frac{Z \Rightarrow Y \mathbin{\check{\odot}} X}{Y \mathbin{\hat{\rightarrow}} Z \Rightarrow X}$$

- Logical Rules ???
  (the following naive proposal is problematic)

$$\frac{A \mathbin{\hat{\oplus}} B \Rightarrow X}{A \oplus B \Rightarrow X} \qquad \frac{X \Rightarrow A \mathbin{\check{\odot}} B}{X \Rightarrow A \odot B}$$

$$\frac{A \mathbin{\hat{\rightarrow}} B \Rightarrow X}{A \rightarrow B \Rightarrow X} \qquad \frac{X \Rightarrow B \mathbin{\check{\ominus}} A}{X \Rightarrow B \ominus A}$$

Ł3 is sound and structures are displayable

We use ALBA (specialized to regular operators) to generate the rule Ł3:

$$\text{Ł3}\ \frac{X_1 \Rightarrow Y_1 \qquad X_2 \Rightarrow Y_2 \qquad X_2 \Rightarrow Y_3}{(X_1 \check{\rightarrow} Y_2) \hat{\rightarrow} X_2 \Rightarrow Y_1 \check{\vee} Y_3}$$

$$\text{Ł3}\ \frac{X_1 \Rightarrow Y_1 \qquad X_2 \Rightarrow Y_2 \qquad X_2 \Rightarrow Y_3}{(X_1 \hat{\ominus} Y_2) \hat{\oplus} X_2 \Rightarrow Y_1 \check{\vee} Y_3}$$

Modulo additional structural rules, we have
$(X_1 \check{\rightarrow} Y_2) \hat{\rightarrow} X_2 = (X_1 \hat{\ominus} Y_2) \hat{\oplus} X_2$.

Assume $x_1 \leq y_1$, $x_2 \leq y_2$ and $x_2 \leq y_3$. Then, the following hold:

1. $(x_1 \ominus y_2) \oplus x_2 \leq y_3 \vee y_1$,
2. $x_2 \leq (y_3 \vee y_1) \ominus (x_1 \ominus y_2)$,
3. $(x_1 \ominus y_2) \leq (y_3 \vee y_1) \ominus x_2$.

Relativized display property: Every structure occurring in a D.Ł-derivable sequent is displayable.

Proof

If $(x_1 \ominus y_2) = 0$ then the first two inequalities are equivalent to $x_2 \leq y_1 \vee y_3$, which follows from $x_2 \leq y_3$ and the third is trivially true. So, let's assume that $(x_1 \ominus y_2) > 0$.

1. From $(x_1 \ominus y_2) > 0$ and $x_2 \leq y_2$ it follows $(x_1 \ominus y_2) \oplus x_2 \leq x_1$ holds. Since $x_1 \leq y_1$ it follows that $x_1 \leq y_1 \vee y_3$ which implies that $(x_1 \ominus y_2) \oplus x_2 \leq y_3 \vee y_1$ holds.

2. We work in cases.
   $(x_1 \ominus y_2) \oplus x_2 < 1$: Then $(x_1 \ominus y_2) \oplus x_2 = (x_1 \ominus y_2) + x_2$. Therefore, from (1), $(x_1 \ominus y_2) + x_2 \leq y_3 \vee y_1$. Hence

   $$x_2 \leq (y_3 \vee y_1) - (x_1 \ominus y_2) \leq (y_3 \vee y_1) \ominus (x_1 \ominus y_2).$$

   $(x_1 \ominus y_2) \oplus x_2 = 1$: Since $(x_1 \ominus y_2) \oplus x_2 \leq x_1$ we have that $x_1 = 1 = y_1$. Then $(y_3 \vee y_1) \ominus (x_1 \ominus y_2) = 1 \ominus (1 \ominus y_2) = y_2$. Hence

   $$x_2 \leq y_2 = (y_3 \vee y_1) \ominus (x_1 \ominus y_2).$$

3. Finally, $x_1 \leq y_3 \vee y_1$ and $x_2 \leq y_2$ imply by the tonicity of $\ominus$ that $(x_1 \ominus y_2) \leq (y_3 \vee y_1) \ominus x_2$.

Deriving Ł3

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B \qquad B \Rightarrow B}{(A \mathbin{\hat{\ominus}} B) \mathbin{\hat{\oplus}} B \Rightarrow A \mathbin{\check{\vee}} B}\text{Ł3}}{(A \mathbin{\hat{\ominus}} B) \mathbin{\hat{\oplus}} B \Rightarrow A \vee B}}{A \mathbin{\hat{\ominus}} B \Rightarrow (A \vee B) \mathbin{\check{\ominus}} B}}{A \ominus B \Rightarrow (A \vee B) \mathbin{\check{\ominus}} B}}{(A \ominus B) \mathbin{\hat{\oplus}} B \Rightarrow A \vee B}
$$

Conclusions

- √ Division of labour between logical and structural rules, modularity, and uniformity.
- Generalize the Belnap's conditions defining (proper) display calculi as to capture regular operators and show canonical cut-elimination.
- Multi-type presentation of Łukasiewicz logic?

References

- A. Ciabattoni, R. Ramanayake, ***Power and limits of structural display rules***, 2016.
- G. Greco, M. Ma, A. Palmigiano, A. Tzimoulis, Z. Zhao, ***Unified correspondence as a proof-theoretic tool***, 2016.
- S. Frittella, G. Greco, A. Kurz, A. Palmigiano, V. Sikimić, ***Multi-type sequent calculi***, 2014.
- W. Fussner, M. Gehrke, S. van Gool, V. Marra, ***Priestley duality of MV-algebras and beyond***, 2021.
- G. Metcalfe, N. Olivetti, D. Gabbay, ***Proof theory for fuzzy logics***, 2009.
- S. Balco, G. Greco, A. Kurz, A. Moshier, A. Palmigiano, A. Tzimoulis, ***First order logic properly displayed***, arxiv.