Syntactic completeness of proper display calculi

Giuseppe Greco
Vrije Universiteit

Joint work with:

Jinsheng Chen, Alessandra Palmigiano & Apostolos Tzimoulis

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Everybody needs somebody

$$\frac{\text{Everybody}}{s \,/\, (np \backslash s)} \otimes \frac{\text{needs}}{(((np \backslash s) \,/\, ((s \,/\, np) \backslash s))} \otimes \frac{\text{somebody}}{(s \,/\, np) \backslash s} \overset{\text{is}}{\vdash} \frac{\text{a s.}}{\mathrm{s}}$$

There are 7 different sequent derivations, but only 3 different natural deduction (or proof net) derivations (in normal form).

Moving to a focused sequent system we have again 3 derivations in normal form.

Different derivations correspond to different readings (where different derivations have different axiom linking):

- Everybody > somebody > needs
- Somebody > everybody > needs

Everybody needs somebody

$$\frac{\text{Everybody}}{s \ / \ (np \ \backslash \ s)} \otimes \frac{\text{needs}}{(((np \ \backslash \ s) \ / \ ((s \ / \ np) \ \backslash \ s))} \otimes \frac{\text{somebody}}{(s \ / \ np) \ \backslash \ s} \overset{\text{is}}{\vdash} \frac{\text{a s.}}{\mathrm{s}}$$

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- Somebody > everybody > needs
- Everybody > needs > somebody

Structural proof theory and automatic rule generation

Structural proof theory focuses on **analytic calculi**: those calculi supporting a *robust form of cut elimination*.

A derivation is **analytic** when all needed information is already contained in its premises and conclusions.

Sequent calculi: inference rules preserving cut elimination can be understood as *analytic rules*.

Automatic rule generation: characterization of classes of axioms corresponding to analytic rules + generation algorithm
(unified correspondence & inverse unified correspondence).

Relational calculi

Some references: the list is not exhaustive!

- Sara Negri and Jan Von Plato. 1998. *Cut elimination in the presence of axioms*

Axioms-as-rules methodology: transforming *universal axioms* in the language of first order classical (or intuitionistic) logics into analytic rules. The rules are used to expand **G3c**.

- Sara Negri. 2003. *Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem*.

Generalization to **geometric implications**, i.e. first order formulas of the form $\forall \overline{z}(A \to B)$ where A and B are *geometric formulas* (i.e. first-order formulas not containing → or ∀).

- Sara Negri. 2005. *Proof analysis in modal logic*.

Application to modal logic axioms: the rules are used to expand **G3K**.

Geometric formulas were first identified and made relevant for proof theory in the context of **natural deduction calculi** in:

- Alex K. Simpson. 1994. *The proof theory and semantics of intuitionistic modal logic*. (Ph.D. Dissertation)
- Luca Vigan00. *Labelled non-classical logics*. (book)

Hypersequent calculi

Some references: the list is not exhaustive!

- Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui. 2008. *From Axioms to Analytic Rules in Nonclassical Logics*.

Substructural hierarchy: A hierarchy of classes of substructural formulas is defined. Substructural axioms up to level $\mathcal{N}_2$ of this hierarchy can be **algorithmically** translated into equivalent rules of a **Gentzen-style sequent calculus**, and axioms up to a subclass of level $\mathcal{P}_3$ into equivalent rules of a **hypersequent calculus**.

- Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui. 2012. *Algebraic proof theory for substructural logics: cut-elimination and completions*.

Generalization to multi-conclusions hypersequents + heuristic to go beyond $\mathcal{P}_3$ axioms.

Display calculi

Some references: the list is not exhaustive!

- Marcus Kracht. 1996. *Power and Weakness of the Modal Display Calculus*.

Primitive axioms in the language of classical tense modal logic can be equivalently captured as analytic structural rules extending the minimal **display calculus for tense logic**.

- Agata Ciabattoni and Revantha Ramanayake. 2016. *Power and Limits of Structural Display Rules*. [CR 16]

Analogous characterization is provided in a more general setting for a given but not fixed **display calculus**, by a procedure for transforming axioms into analytic structural rules (and showing the converse direction whenever the calculus satisfies additional conditions).

- Giuseppe Greco, Minghui Ma, Alessandra Palmigiano, Apostolos Tzimoulis, and Zhiguang Zhao. 2018. *Unified correspondence as a proof-theoretic tool*. [G. et al. 18]

Analogous characterization for arbitrary normal (D)LE-logics via connection with generalized Sahlqvist **correspondence theory**: **ALBA** (which computes the first-order correspondent of (analytic) inductive (D)LE-axioms) can be used to compute analytic rule(s) too.

Analytic inductive ↔ analytic rules ↔ geometric formulas

$$\begin{tikzcd}
 & & & {[\,\mathrm{LR}\,]_{\dashv\vdash}} & & \\
\text{modal formulas} \arrow[r, phantom, "\supseteq"] & \text{analytic inductive} \arrow[r, phantom, "\ni"] \arrow[rrr, leftrightarrow, bend left=60] & \varphi \arrow[r, leftrightarrow, "\text{FO correspondent}", "\text{modal companion}"'] \arrow[ur, leftrightarrow, bend left, "\text{MASSA}"] \arrow[dr, leftrightarrow, bend right, "\text{ALBA}"'] & \psi \arrow[u, leftrightarrow] \arrow[d, leftrightarrow] \arrow[r, phantom, "\in"] & \text{geom. formula} \arrow[r, phantom, "\subseteq"] & \text{first-order formulas} \\
 & & & {[\,\mathrm{DR}\,]_{\dashv\vdash}} & &
\end{tikzcd}$$

Our contributions

The **semantic** equivalence between each **analytic inductive axiom** $\varphi \vdash \psi$ and its corresponding **analytic structural rule(s)** $R_1, \ldots, R_n$ is an immediate consequence of the soundness of the rules of ALBA on perfect normal (distributive) lattice expansions (see [G. et al. 18]).

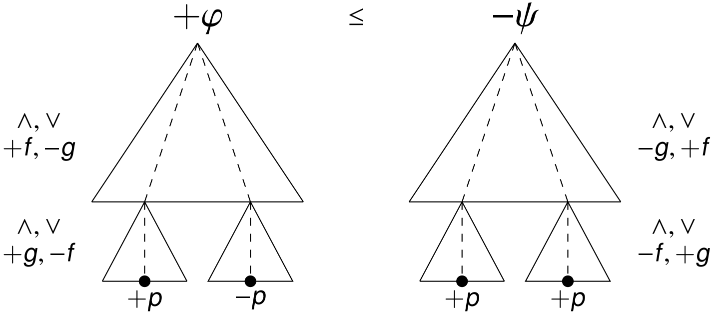
On the **syntactic** side, a description of the derivation, which relies on the proof-theoretic version of Ackermann's Lemma and therefore involves *cuts*, is presented in [CR 16].

An **effective procedure** P was still missing for building *cut-free* derivations of $\varphi \vdash \psi$ in the basic proper display calculus D.(D)LE expanded with $R_1, \ldots, R_n$.

P establishes, via syntactic means, that:

- for any properly displayable (D)LE-logic L, the proper display calculus $D.L$ derives all the theorems (or derivable sequents) of L: **syntactic completeness**.
- P generate a *cut-free* derivation of a particular shape we call in **pre-normal form**.

Analytic-inductive inequalities



Analytic inductive ⇒ Inductive ⇒ Canonical

Examples

The definition of **analytic inductive inequalities** is uniform in each signature.

- Analytic inductive axioms

  $(A \rightarrow B) \vee (B \rightarrow A)$

  $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$

- Sahlqvist but non-analytic axioms

  $A \rightarrow \Diamond\Box A$

  $(\Box A \rightarrow \Diamond B) \rightarrow (A \rightarrow B)$

Basic normal LE-logics and associated display calculi

We define the proper display calculus *D.LE* for the basic normal $\mathcal{L}_{LE}$-logic in a fixed but arbitrary LE-signature $\mathcal{L} = \mathcal{L}(\mathcal{F}, \mathcal{G})$.

Let $S_{\mathcal{F}} := \{\hat{f} \mid f \in \mathcal{F}^*\}$ and $S_{\mathcal{G}} := \{\check{g} \mid g \in \mathcal{G}^*\}$ be the sets of structural connectives associated with $\mathcal{F}^*$ and $\mathcal{G}^*$ respectively (**fully residuated signature**).

Each such structural connective comes with an **arity** and an **order-type** which coincides with those of its associated operational connective in $\mathcal{F}^*$ and $\mathcal{G}^*$.

Theorem

The logic L_{LE}^* is a ***conservative extension*** *of L_{LE}, i.e. every $\mathcal{L}_{LE}$-sequent $\varphi \vdash \psi$ is derivable in L_{LE} if and only if $\varphi \vdash \psi$ is derivable in L_{LE}^* (we use* ***canonical extensions****).*

Display calculi for basic normal LE-logics

The calculus $D.LE$ manipulates sequents $\Pi \vdash \Sigma$ where the structures Π (for precedent) and Σ (for succedent) are defined by the following simultaneous recursion:

$$Str_{\mathcal{F}} \ni \Pi ::= \varphi \mid \hat{\top} \mid \hat{f}\,(\overline{\Pi}^{(\varepsilon_f)})$$
$$Str_{\mathcal{G}} \ni \Sigma ::= \varphi \mid \check{\bot} \mid \check{g}\,(\overline{\Sigma}^{(\varepsilon_g)})$$

For any connective h of arity $n \geq 1$, the notational convention

- $\hat{h}$ conveys also the information that h is a **left-adjoint/residual**
- $\check{h}$ conveys the information that h is a **right-adjoint/residual**

We use $\Upsilon_1, \ldots, \Upsilon_n$ as structure metavariables in $Str_{\mathcal{F}} \cup Str_{\mathcal{G}}$.

The introduction rules of the calculus below will guarantee that:

- $\Upsilon \in Str_{\mathcal{F}}$ whenever it occurs in **precedent position**
- and $\Upsilon \in Str_{\mathcal{G}}$ whenever it occurs in **succedent position**

Lattice reduct

- Identity and cut rules:

$$\mathrm{Id}\ \frac{}{p \vdash p} \qquad \frac{\Pi \vdash \varphi \qquad \varphi \vdash \Sigma}{\Pi \vdash \Sigma}\ \mathrm{Cut}$$

- Structural rules for lattice connectives:

$$\top_W\ \frac{\hat{\top} \vdash \Sigma}{\Pi \vdash \Sigma} \qquad \frac{\Pi \vdash \check{\bot}}{\Pi \vdash \Sigma}\ \bot_W$$

- Logical introduction rules for lattice connectives:

$$\top_L\ \frac{\hat{\top} \vdash \Sigma}{\top \vdash \Sigma} \qquad \frac{}{\hat{\top} \vdash \top}\ \top_R \qquad \bot_L\ \frac{}{\bot \vdash \check{\bot}} \qquad \frac{\Pi \vdash \check{\bot}}{\Pi \vdash \bot}\ \bot_R$$

$$\wedge_{L2}\ \frac{\psi \vdash \Sigma}{\varphi \wedge \psi \vdash \Sigma} \qquad \wedge_{L1}\ \frac{\varphi \vdash \Sigma}{\varphi \wedge \psi \vdash \Sigma} \qquad \frac{\Pi \vdash \varphi \qquad \Pi \vdash \psi}{\Pi \vdash \varphi \wedge \psi}\ \wedge_R$$

$$\vee_L\ \frac{\varphi \vdash \Sigma \qquad \psi \vdash \Sigma}{\varphi \vee \psi \vdash \Sigma} \qquad \frac{\Pi \vdash \varphi}{\Pi \vdash \varphi \vee \psi}\ \vee_{R1} \qquad \frac{\Pi \vdash \psi}{\Pi \vdash \varphi \vee \psi}\ \vee_{R2}$$

Display postulates for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

- for any $1 \leq i, j \leq n_f$ and $1 \leq h, k \leq n_g$,

  If $\varepsilon_f(i) = 1$ and $\varepsilon_g(h) = 1$,

$$\hat{f} \dashv \check{f}_i^\sharp \; \frac{\hat{f}\,(\Upsilon_1, \ldots, \Pi_i, \ldots, \Upsilon_{n_f}) \vdash \Sigma}{\Pi_i \vdash \check{f}_i^\sharp\,(\Upsilon_1, \ldots, \Sigma, \ldots, \Upsilon_{n_f})} \quad \frac{\Pi \vdash \check{g}\,(\Upsilon_1 \ldots, \Sigma_h, \ldots \Upsilon_{n_g})}{\hat{g}_h^\flat\,(\Upsilon_1, \ldots, \Pi, \ldots, \Upsilon_{n_g}) \vdash \Sigma_h} \; \hat{g}_h^\flat \dashv \check{g}$$

  If $\varepsilon_f(j) = \partial$ and $\varepsilon_g(k) = \partial$,

$$(\hat{f}, \hat{f}_j^\sharp) \; \frac{\hat{f}\,(\Upsilon_1, \ldots, \Sigma_j, \ldots, \Upsilon_{n_f}) \vdash \Sigma}{\hat{f}_j^\sharp\,(\Upsilon_1, \ldots, \Sigma, \ldots, \Upsilon_{n_f}) \vdash \Sigma_j} \quad \frac{\Pi \vdash \check{g}\,(\Upsilon_1, \ldots, \Pi_k, \ldots, \Upsilon_{n_g})}{\Pi_k \vdash \check{g}_k^\flat\,(\Upsilon_1, \ldots, \Pi, \ldots, \Upsilon_{n_g})} \; (\check{g}, \check{g}_k^\flat)$$

Logical rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

We omit the rules for a generic connective $h \in (\mathcal{F} \cap \mathcal{G})$ of arity $n = 1$.

$$\frac{\Big(\Upsilon_i \vdash \varphi_i \quad \varphi_j \vdash \Upsilon_j \mid 1 \leq i, j \leq n_f, \varepsilon_f(i) = 1 \text{ and } \varepsilon_f(j) = \partial \Big)}{\hat{f}\,(\Upsilon_1, \ldots, \Upsilon_{n_f}) \vdash f(\varphi_1, \ldots, \varphi_{n_f})}\ f_R$$

$$g_L\ \frac{\Big(\varphi_i \vdash \Upsilon_i \quad \Upsilon_j \vdash \varphi_j \mid 1 \leq i, j \leq n_g, \varepsilon_g(i) = 1 \text{ and } \varepsilon_g(j) = \partial \Big)}{g(\varphi_1, \ldots, \varphi_{n_g}) \vdash \check{g}\,(\Upsilon_1, \ldots, \Upsilon_{n_g})}$$

$$f_L\ \frac{\hat{f}\,(\varphi_1, \ldots, \varphi_{n_f}) \vdash \Sigma}{f(\varphi_1, \ldots, \varphi_{n_f}) \vdash \Sigma} \qquad \frac{\Pi \vdash \check{g}\,(\varphi_1, \ldots, \varphi_{n_g})}{\Pi \vdash g(\varphi_1, \ldots, \varphi_{n_g})}\ g_R$$

Proposition

*The calculus D.LE (hence also $\underline{\text{D.LE}}$) is **sound** w.r.t. the class of complete $\mathcal{L}$-algebras.*

Proposition

*The calculus D.LE is a proper display calculus, and hence **cut elimination** holds for it as a consequence of a Belnap-style cut elimination meta-theorem.*

Distributive lattice reduct

The language of $D.DLE$ for the basic $\mathcal{L}_{DLE}$-logic is obtained by augmenting the language of $D.LE$ with:

Structural symbols	$\hat{\top}$	$\check{\bot}$	$\hat{\wedge}$	$\check{\vee}$	$\hat{\succ\mkern-8mu-}$	$\check{\rightarrow}$	$\hat{-\mkern-8mu\prec}$	$\check{\leftarrow}$
Operational symbols	$\top$	$\bot$	$\wedge$	$\vee$	$(\succ\mkern-8mu-)$	$(\rightarrow)$	$(-\mkern-8mu\prec)$	$(\leftarrow)$

Since $\wedge$ and $\vee$ distribute over each other, besides being Δ-adjoints, they can also be treated as elements of $\mathcal{F}$ and $\mathcal{G}$ respectively: the **display postulates** and **logical rules** follow the same pattern and we omit them.

The **structural rules** encoding the characterizing properties of the lattice connectives are as expected and we omit them.

Derivations in pre-normal form

If $\varphi \vdash \psi$ is a **definite** analytic inductive axiom (Ax), then ALBA yields a **single** analytic structural rule $R(Ax)$ corresponding to it.

Both in the general and in the distributive settings, the Skeleton part of the derivation of $\varphi \vdash \psi$ in pre-normal form will only have one branch, yielding the following simpler shape of π:

$$
\begin{array}{rcl}
\mathrm{PIA}(\pi)\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. &
\begin{array}{c}
p_{1.1}\vdash p_{1.1} \quad \cdots \quad p_{1.k}\vdash p_{1.k} \quad \cdots \quad p_{n.1}\vdash p_{n.1} \quad \cdots \quad p_{n.\ell}\vdash p_{n.\ell} \\
\ddots\vdots\iddots \qquad\qquad\qquad\qquad\qquad \ddots\vdots\iddots \\
\Pi_1\vdash\Sigma_1 \qquad\qquad\qquad \cdots \qquad\qquad\qquad \Pi_n\vdash\Sigma_n
\end{array} & \\ \cline{2-2}
\mathrm{Skeleton}(\pi)\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. &
\begin{array}{c} \Pi\vdash\Sigma \\ \vdots \\ \varphi\vdash\psi \end{array} & R(Ax)
\end{array}
$$

If $\varphi \vdash \psi$ is **not definite**, it can be equivalently transformed into a set of definite axioms $Ax_1, \ldots, A_n$, each of which will correspond to one analytic structural rule $R(Ax_1), \ldots, R(A_n)$. In this case, the Skeleton part of the derivation is branching.

Lattice

(i) **Skeleton(π)** is the proof-subtree of π containing the root of π and applications of **invertible rules** for the introduction of all connectives occurring in the Skeleton of $\varphi \vdash \psi$ (possibly modulo applications of display rules);

(ii) **PIA(π)** is a collection of proof-subtrees of π containing the initial axioms of π and all the applications of **non-invertible rules** for the introduction of connectives occurring in the maximal PIA-subtrees in the signed generation trees of $\varphi \vdash \psi$ (possibly modulo applications of display rules) and such that

(iii) the root of each proof-subtree in PIA(π) coincides with a premise of the application of $R(Ax)$ in π, where the atomic structural variables are suitably instantiated with maximal PIA-subformulas of $\varphi \vdash \psi$.

Distributive lattice

(i) **Skeleton(π)** is the proof-subtree of π containing, possibly modulo applications of display rules, the root of π and applications of
  (a) **invertible rules** for the introduction of all connectives occurring as SLR nodes ($+f, -g$ with $n \geq 1$) in the Skeleton of $\varphi \vdash \psi$;
  (b) **non-invertible rules** and **Contraction** for the introduction of all connectives occurring as Δ-adjoint nodes ($+\vee, -\wedge$) in the Skeleton of $\varphi \vdash \psi$;

(ii) **PIA(π)** is a collection of proof-subtrees of π containing, possibly modulo applications of display rules, the initial axioms of π and applications of
  (a) **non-invertible rules** for the introduction of all connectives occurring as unary SRA nodes ($+g, -f$) or as SRR nodes ($+\vee, -\wedge$ and $+g, -f$ with $n \geq 2$) in the maximal PIA-subtrees in the signed generation trees of $\varphi \vdash \psi$;
  (b) **invertible rules** and **Weakening** for the introduction of all lattice connectives occurring as SRA nodes ($+\wedge, -\vee$) in the maximal PIA-subtrees in the signed generation trees of $\varphi \vdash \psi$;

(iii) the root of each proof-subtree in PIA(π) coincides with a premise of the application of $R(Ax)$ in π, where the atomic structural variables are suitably instantiated with operational maximal PIA-subtrees of $\varphi \vdash \psi$.

Syntactic completeness

Theorem

Any analytic inductive LE-axiom (resp. DLE-axiom) $\varphi \vdash \psi$ *can be effectively derived in the corresponding basic cut-free calculus* $\underline{\text{D.LE}}$ *(resp.* $\underline{\text{D.DLE}}$*) enriched with the structural analytic rules* $R_1, \ldots, R_n$ *corresponding to* $\varphi \vdash \psi$*.*

Moreover, the cut-free derivation is in pre-normal form.

...to prove the theorem we need a few lemmas

- Two key-lemmas provide the tools for obtaining the sub-derivations in **PIA(π)**. An inspection on the proofs of these results reveals that indeed only non-invertible logical rules and display rules are applied.
- Two key-lemmas provide the tools involving the introduction of the **lattice connectives**. An inspection on the proofs reveals that only introduction rules of one type are applied in each component.

We represent (Ω, ε)-analytic inductive inequalities/sequents as follows:

$$(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}] \qquad (\varphi \vdash \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}],$$

where:

- $(\varphi \leq \psi)[!\overline{x}, !\overline{y}, !\overline{z}, !\overline{w}]$ is the skeleton of the given inequality, $\overline{\alpha}$ (resp. $\overline{\beta}$) denotes the vector of positive (resp. negative) maximal PIA subformulas
- $\overline{\gamma}$ (resp. $\overline{\delta}$) denotes the vector of positive (resp. negative) maximal ε^{∂}-uniform PIA subformulas

Computing the analytic-inductive rule

ALBA-run computing the structural rule for $\Diamond\Box(p \wedge q) \vdash \Box\Diamond p \vee \Box\Diamond q$:

$$\begin{array}{ll}
 & \Diamond\Box(p \wedge q) \vdash \Box\Diamond p \vee \Box\Diamond q \\
\text{iff} & \forall p \forall q \forall x \forall y \forall z\, [\, x \vdash \Box(p \wedge q)\ \&\ \Diamond p \vdash y\ \&\ \Diamond q \vdash z \Rightarrow \Diamond x \vdash \Box y \vee \Box z \,] \\
\text{iff} & \forall p \forall q \forall x \forall y \forall z\, [\, x \vdash \Box(p \wedge q)\ \&\ p \vdash \blacksquare y\ \&\ q \vdash \blacksquare z \Rightarrow \Diamond x \vdash \Box y \vee \Box z \,] \\
\text{iff} & \forall x \forall y \forall z\, [\, x \vdash \Box(\blacksquare y \wedge \blacksquare z) \Rightarrow \Diamond x \vdash \Box y \vee \Box z \,] \\
\text{iff} & \forall x \forall y \forall z\, [\, x \vdash \Box\blacksquare y\ \&\ x \vdash \Box\blacksquare z \Rightarrow \Diamond x \vdash \Box y \vee \Box z \,]
\end{array}$$

Producing the derivation in pre-normal form

$$
\begin{prooftree}
\AxiomC{$p \vdash p$}
\RightLabel{$\Diamond_R$}
\UnaryInfC{$\hat{\Diamond} p \vdash \Diamond p$}
\RightLabel{$\hat{\Diamond} \dashv \check{\blacksquare}$}
\UnaryInfC{$p \vdash \check{\blacksquare} \Diamond p$}
\LeftLabel{W_L}
\UnaryInfC{$p \mathbin{\hat{\wedge}} q \vdash \check{\blacksquare} \Diamond p$}
\LeftLabel{$\wedge_L$}
\UnaryInfC{$p \wedge q \vdash \check{\blacksquare} \Diamond p$}
\LeftLabel{$\Box_L$}
\UnaryInfC{$\Box(p \wedge q) \vdash \check{\Box} \check{\blacksquare} \Diamond p$}
\AxiomC{$q \vdash q$}
\RightLabel{$\Diamond_R$}
\UnaryInfC{$\hat{\Diamond} q \vdash \Diamond q$}
\RightLabel{$\hat{\Diamond} \dashv \check{\blacksquare}$}
\UnaryInfC{$q \vdash \check{\blacksquare} \Diamond q$}
\LeftLabel{W_L}
\UnaryInfC{$q \mathbin{\hat{\wedge}} p \vdash \check{\blacksquare} \Diamond q$}
\LeftLabel{E_L}
\UnaryInfC{$p \mathbin{\hat{\wedge}} q \vdash \check{\blacksquare} \Diamond q$}
\LeftLabel{$\wedge_L$}
\UnaryInfC{$p \wedge q \vdash \check{\blacksquare} \Diamond q$}
\LeftLabel{$\Box_L$}
\UnaryInfC{$\Box(p \wedge q) \vdash \check{\Box} \check{\blacksquare} \Diamond q$}
\RightLabel{R (Lemma)}
\BinaryInfC{$\hat{\Diamond} \Box(p \wedge q) \vdash \check{\Box} \Diamond p \mathbin{\check{\vee}} \check{\Box} \Diamond q$}
\LeftLabel{$\hat{-\!\!<} \dashv \check{\vee}$}
\UnaryInfC{$\hat{\Diamond} \Box(p \wedge q) \mathbin{\hat{-\!\!<}} \check{\Box} \Diamond q \vdash \check{\Box} \Diamond p$}
\RightLabel{$\Box_R$}
\UnaryInfC{$\hat{\Diamond} \Box(p \wedge q) \mathbin{\hat{-\!\!<}} \check{\Box} \Diamond q \vdash \Box \Diamond p$}
\RightLabel{$\hat{-\!\!<} \dashv \check{\vee}$}
\UnaryInfC{$\hat{\Diamond} \Box(p \wedge q) \vdash \Box \Diamond p \mathbin{\check{\vee}} \check{\Box} \Diamond q$}
\LeftLabel{$\hat{>\!\!-} \dashv \check{\vee}$}
\UnaryInfC{$\Box \Diamond p \mathbin{\hat{>\!\!-}} \hat{\Diamond} \Box(p \wedge q) \vdash \check{\Box} \Diamond q$}
\RightLabel{$\Box_R$}
\UnaryInfC{$\Diamond p \mathbin{\hat{>\!\!-}} \hat{\Diamond} \Box(p \wedge q) \vdash \Box \Diamond q$}
\RightLabel{$\hat{>\!\!-} \dashv \check{\vee}$}
\UnaryInfC{$\hat{\Diamond} \Box(p \wedge q) \vdash \Box \Diamond p \mathbin{\check{\vee}} \Box \Diamond q$}
\RightLabel{$\vee_R$}
\UnaryInfC{$\hat{\Diamond} \Box(p \wedge q) \vdash \Box \Diamond p \vee \Box \Diamond q$}
\LeftLabel{$\Diamond_L$}
\UnaryInfC{$\Diamond \Box(p \wedge q) \vdash \Box \Diamond p \vee \Box \Diamond q$}
\end{prooftree}
$$

(Lemma)

Conclusions

- ✓ Proper display calculi enjoys syntactic completeness and derivations in pre normal form can be effectively produced.
- ► Provide formal translations between derivations in different formalisms.