Projective unification through duality

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Solving logical equations

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Possible solutions: $\mathbf{x} := \neg p$, $\mathbf{x} := \top$, ...

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$\mathbf{x} := p \wedge \langle \text{Tomorrow} \rangle \neg p$ is a solution.

But also
$\mathbf{x} := p \wedge q \wedge \langle \text{Tomorrow} \rangle \neg p$
$\mathbf{x} := p \wedge q \wedge r \wedge \langle \text{Tomorrow} \rangle \neg p$
$\ldots$

How to describe the set of solutions?

The problem of unification

The modal language $\mathcal{L}_P$ is defined by

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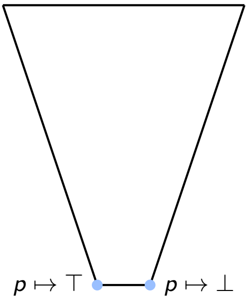
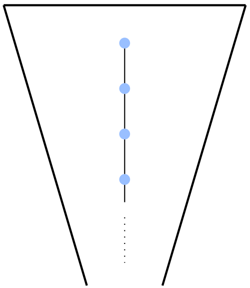
We write $\sigma \preceq_{\mathbf{L}} \tau$ whenever $\tau \equiv_{\mathbf{L}} \mu \circ \sigma$ for some substitution μ.
$\sigma \preceq_{\mathbf{L}} \tau$ reads "σ is at least as general as τ".

Structural concerns

How nice unification is in $\mathbf{L}$ depends on the properties of $\preceq_{\mathbf{L}}$.



Unifiers of $p \to \Box p$ in $\mathbf{K} + \Diamond\top$



Unifiers of $p \to \Box p$ in $\mathbf{K}$
(Jeřábek 2015)

Projective unification

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Example

The substitution σ defined by $\sigma(p) := p \wedge \Box\neg p$ is a projective unifier of $p \rightarrow \Box\neg p$ in $\mathbf{K}$.

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Definition

A logic **L** is projective if every unifiable formula possesses a projective unifier.

The logics **K45**, **S4.3** and **S5** are projective. There are not many examples...

Duality

We denote by $\mathbf{A}_P$ the Lindenbaum algebra of $\mathbf{L}$ over the variables in P. A substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ can be identified to a homomorphism $\sigma : \mathbf{A}_P \to \mathbf{A}_Q$.

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$$\begin{array}{ccc} \mathbf{A}_P & \xrightarrow{\quad\sigma\quad} & \mathbf{A}_Q \\ \vdots & & \vdots \\ \mathfrak{F}_P & \xleftarrow{\quad\sigma^*\quad} & \mathfrak{F}_Q \end{array}$$

Here $\mathfrak{F}_P$ is the canonical Kripke frame over P.

Dual unifiers

Let $\widehat{\varphi}$ denote the extension of a formula $\varphi \in \mathcal{L}_P$ within $\mathfrak{F}_P$. We then define the tight extension of φ as

$$\widehat{\varphi}^{\infty} := \bigcap_{n \in \mathbb{N}} \widehat{\Box^n \varphi}.$$

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A dual unifier of $\varphi \in \mathcal{L}_P$ is a map $f : \mathfrak{F}_Q \to \mathfrak{F}_P$ such that:

1. f is a bounded morphism;
2. for all $\psi \in \mathcal{L}_P$ there exists $\theta \in \mathcal{L}_Q$ such that $f^{-1}[\widehat{\psi}] = \widehat{\theta}$ (*continuity*);
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Theorem

σ is a unifier of φ iff σ^* is a dual unifier of φ.

Projective dual unifiers

A projective dual unifier of φ is a dual unifier $f : \mathfrak{F}_P \to \mathfrak{F}_P$ of φ such that

$$f(x) = x \text{ for all } x \in \widehat{\varphi}^{\infty}.$$

Theorem

σ is a projective unifier of φ iff σ^* is a projective dual unifier of φ.

Application to $\mathbf{K4}_n\mathbf{B}_k$

All extensions of
$\mathbf{K4}_n\mathbf{B}_k := \mathbf{K} + (\Box^{\leq n} p \to \Box^{n+1} p) + (p \to \Box^{\leq k} \Diamond^{\leq k} p)$ are known to be projective (Kostrzycka, 2022).
We propose a proof based on duality.

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Proof sketch.
We fix $\varphi \in \mathcal{L}_P$.
Since $\vdash_{\mathbf{L}} p \to \Box^{\leq k}\Diamond^{\leq k} p$ the frame $\mathfrak{F}_P = (X, R)$ is *k-symmetric*:

$$xR^{\leq k}y \implies yR^{\leq k}x$$

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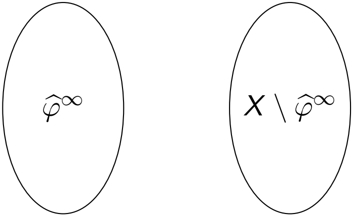
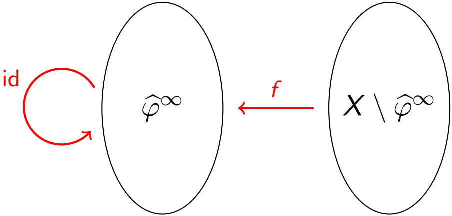
Hence $\widehat{\varphi}^{\infty} := \bigcap_{n \in \mathbb{N}} \widehat{\Box^n \varphi}$ is both upward closed and downward closed (with respect to R).

Application to $\mathbf{K4}_n\mathbf{B}_k$



Suppose that φ has a dual unifier $f : \mathfrak{F}_P \to \mathfrak{F}_P$ in $\mathbf{L}$. ⚠



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Define $g(x) := \begin{cases} x & \text{if } x \in \widehat{\varphi}^\infty \\ f(x) & \text{otherwise} \end{cases}$.

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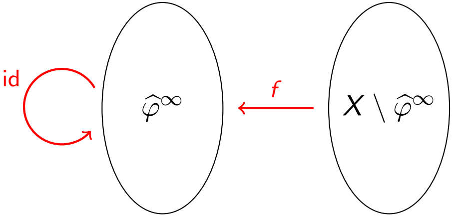

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Define $g(x) := \begin{cases} x & \text{if } x \in \widehat{\varphi}^{\infty} \\ f(x) & \text{otherwise} \end{cases}$.

$\mathrm{Im}(g) \subseteq \widehat{\varphi}^{\infty}$ ✓ (since $\mathrm{Im}(f) \subseteq \widehat{\varphi}^{\infty}$)

g bounded morphism ✓ (since f bounded morphism)

$g(x) = x$ for all $x \in \widehat{\varphi}^{\infty}$ ✓

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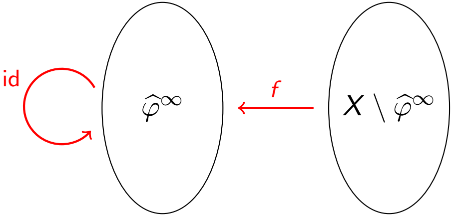

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Continuity: $\vdash_{\mathbf{L}} \Box^{\leq n} p \to \Box^{n+1} p$ yields $\widehat{\varphi}^{\infty} := \bigcap_{n \in \mathbb{N}} \widehat{\Box^n \varphi} = \widehat{\Box^{\leq n} \varphi}$,

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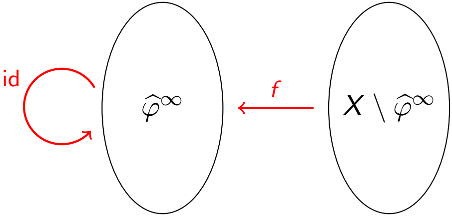

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$$\begin{aligned} g^{-1}[\widehat{\psi}] &= (\widehat{\psi} \cap \widehat{\varphi}^\infty) \cup (f^{-1}[\widehat{\psi}] \cap X \setminus \widehat{\varphi}^\infty) \\ &= (\widehat{\psi} \cap \widehat{\Box^{\leq n} \varphi}) \cup (f^{-1}[\widehat{\psi}] \cap \widehat{\neg \Box^{\leq n} \varphi}). \end{aligned}$$

Application to **K4D1**

Kost (2018) showed that the projective extensions of **K4** are exactly the extensions of

$$\mathbf{K4D1} := \mathbf{K4} + \Box(\Box p \to q) \vee \Box(\Box q \to p).$$

We partially recover this result.

Definition

A logic $\mathbf{L}$ is locally tabular if $\mathbf{A}_P$ is finite for all finite P.

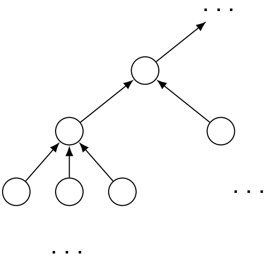
Theorem

If $\mathbf{K4D1} \subseteq \mathbf{L}$ *and* $\mathbf{L}$ *is locally tabular then* $\mathbf{L}$ *is projective.*

Application to **K4D1**

Proof sketch. Since **K4D1** $\subseteq$ **L**, the frame $\mathfrak{F}_P = (X, R)$ is transitive and *linear*:

$$xRy \text{ and } xRz \implies yRz \text{ or } zRy$$



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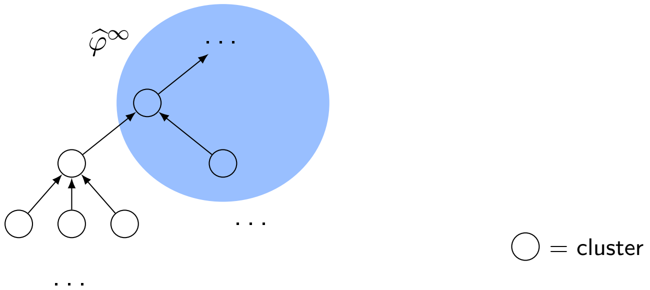

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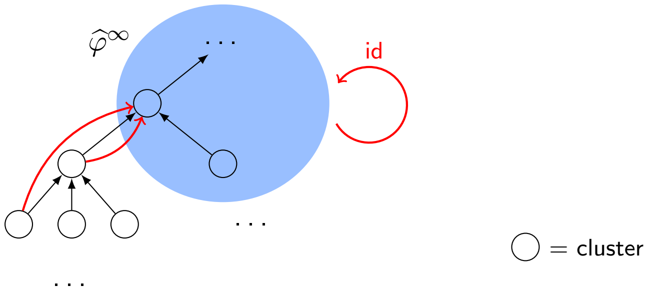

Suppose that φ has a dual unifier $f : \mathfrak{F}_P \to \mathfrak{F}_P$ in **L**.
We define

$$g(x) := \begin{cases} x & \text{if } x \in \widehat{\varphi}^{\infty} \\ \text{some } R\text{-mininal } y \in \widehat{\varphi}^{\infty} \text{ s.t. } xRy & \text{otherwise, if such } y \text{ exists} \\ f(x) & \text{otherwise} \end{cases}$$

The projective extensions of **K5**

The projective character of **K5** used to be unknown.

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Theorem

If **K5** $\subseteq$ **L** *but* **K45** $\not\subseteq$ **L** *then* $\Diamond\Diamond p \to \Diamond p$ *is unifiable but not projective in* **L**.

$$\underbrace{\mathbf{K5} \qquad\qquad\qquad}_{\text{not projective}}\ \underbrace{\mathbf{K45} \qquad \cdots}_{\text{projective}}$$

The projective extensions of $\mathbf{K4}_n\mathbf{D1}_n$

We write

$$\mathbf{K4}_n := \mathbf{K} + \Box^{\leq n} p \to \Box^{n+1} p,$$
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Theorem

If $\mathbf{K4}_n \subseteq \mathbf{L}$ and $\mathbf{K4}_n\mathbf{D1}_n \not\subseteq \mathbf{L}$ then $\Box(\Box^{\leq n} p \to q) \vee \Box(\Box^{\leq n} q \to p)$ is unifiable but not projective in $\mathbf{L}$.

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Future work

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Then σ is an unifier of φ relatively to θ:

$$\theta \vdash_{\mathbf{L}} \sigma(\varphi).$$

Thanks for listening!