Jean Goubault-Larrecq

Noetherian spaces, wqos, and their statures

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With Bastien Laboureix, Aliaume Lopez, Simon Halfon

Laboratoire Méthodes Formelles

université PARIS-SACLAY

cnrs

école normale supérieure paris—saclay

Outline

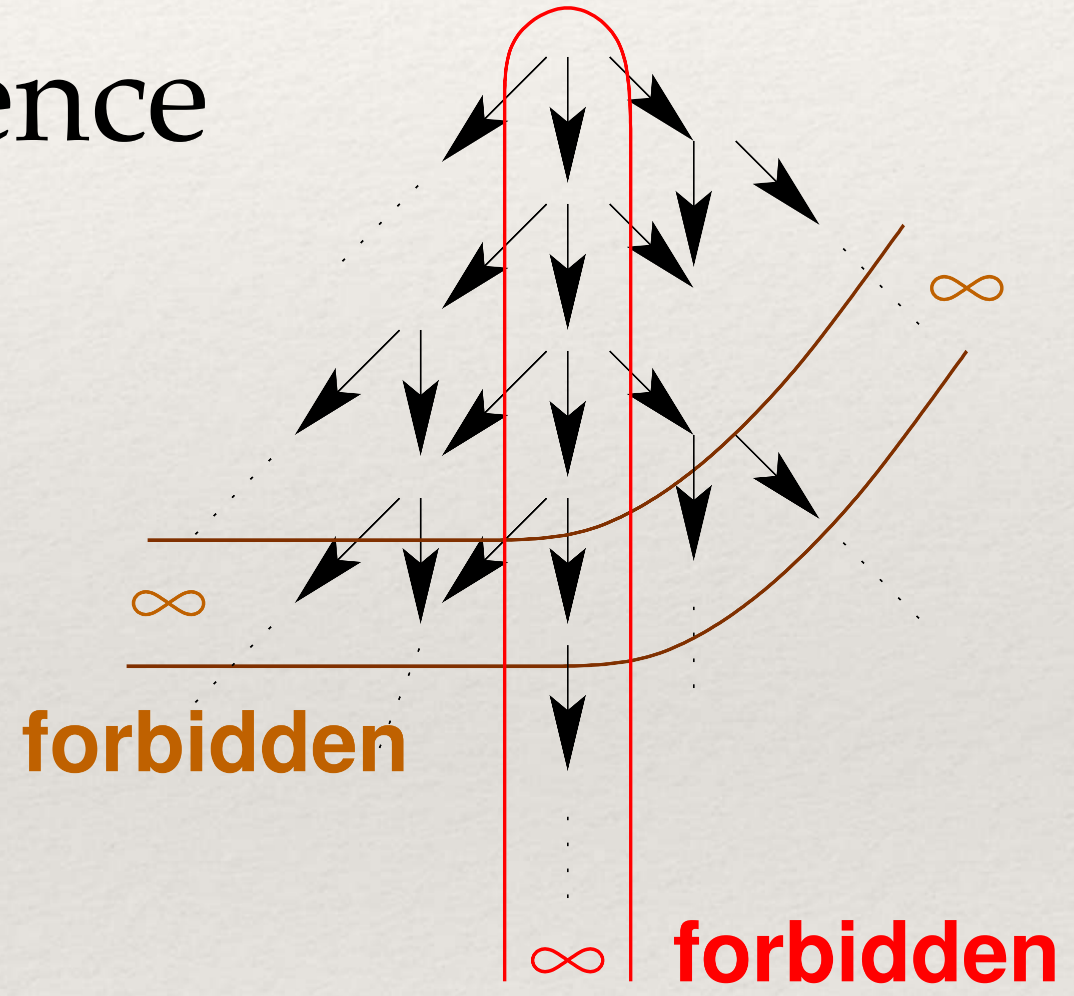
- Noetherian spaces and wqos
- A computer scientist's view
- Sobrifications of Noetherian spaces, and their representations
- Statures of Noetherian spaces and maximal order types of wqos

Noetherian spaces and wqos

Well-quasi-orders

- **Fact.** The following are equivalent for a quasi-ordering $\leq$:
  (1) Every sequence $(x_n)_{n\in\mathbb{N}}$ is **good**: $x_m \leq x_n$ for some $m < n$
  (2) Every sequence $(x_n)_{n\in\mathbb{N}}$ is **perfect**: has a monotone subsequence
  (3) $\leq$ is **well-founded** and has **no infinite antichain**.

- **Defn.** Such a quasi-ordering $\leq$ is called a **well-quasi-order (wqo).**



- Applications:
  classification of graphs (Kuratowski, Robertson-Seymour)
  verification (computer science)
  model theory (logic: Fraïssé, Jullien, Pouzet)

Examples

- $\mathbb{N}$, with its usual ordering — More generally, any **total well-founded** order
- Every **finite** set, with any quasi-ordering
- Finite disjoint **sums**, finite **products** of wqos are wqo
- **Images** of wqos by monotonic maps are wqo (in particular quotients)
- **Inverse images** of wqos by order-reflecting maps are wqo (in particular subsets)
- **Higman's Lemma.** Let $X^* = \{$finite words over alphabet $X\}$ ordered by word embedding $\leq_*$. Then X wqo $\Leftrightarrow X^*$ wqo
- **Kruskal's Theorem.** Let $\mathscr{T}(X) = \{$finite trees with X-labeled vertices$\}$ ordered by homeomorphic tree embedding $\leq_T$. Then X wqo $\Leftrightarrow \mathscr{T}(X)$ wqo.
- And so on.

A computer scientist's view

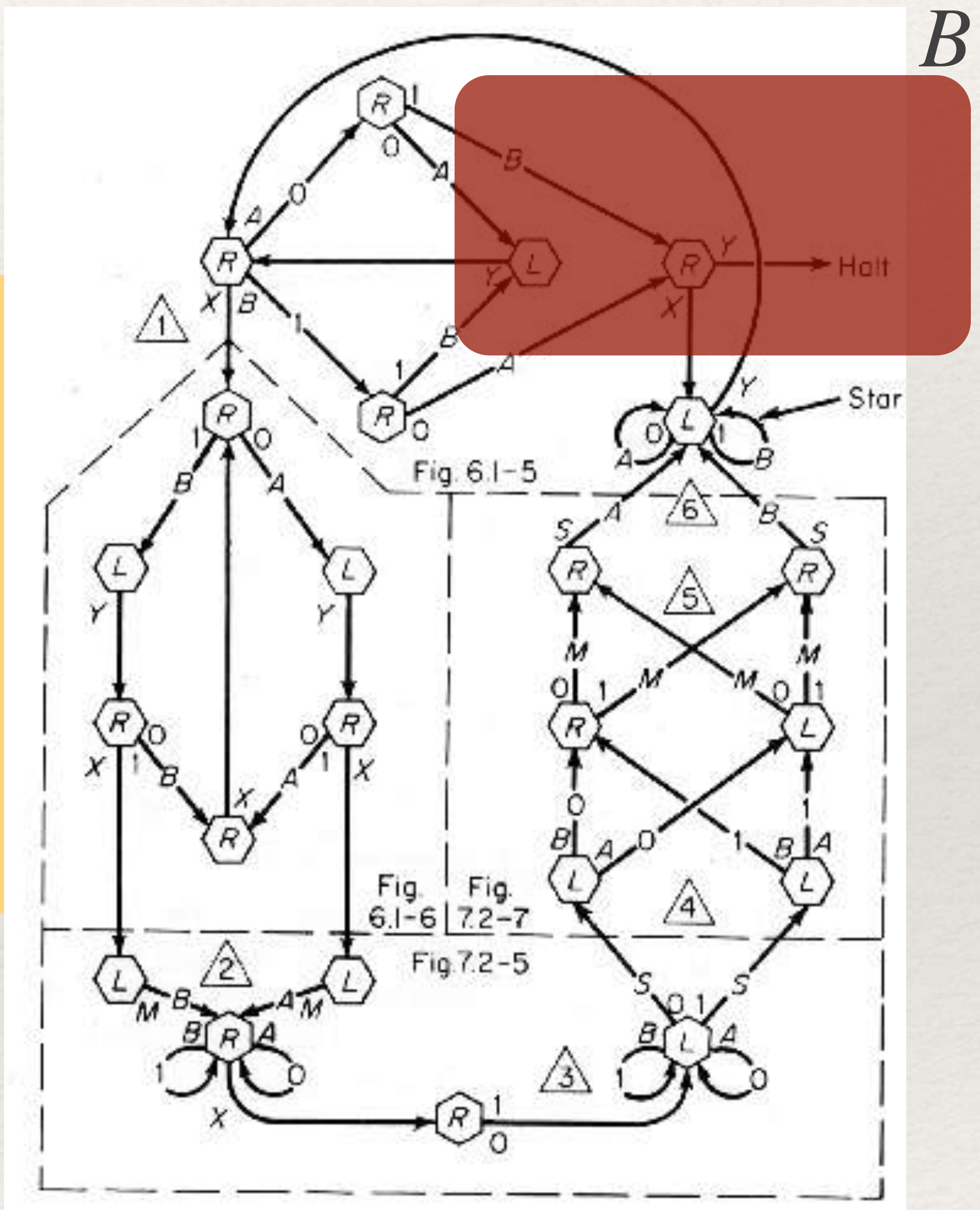
Transition systems

- A **transition system** is just a directed graph (not necessarily finite)
  Vertices are **configurations** (of a computer system, say)
  Computation proceeds in steps $C \to C'$ (along edges)

- **Reachability:** Given a starting configuration C_0,
  and a set B of configurations,
  can one reach B from C_0?
  (i.e., is there a $C \in B$ such that $C_0 \to^* C$?)

- Decidable for finite transition systems (in polynomial time)

- In general **undecidable**: consider the graph of configurations
  of a universal Turing machine, B = {accepting configurations}

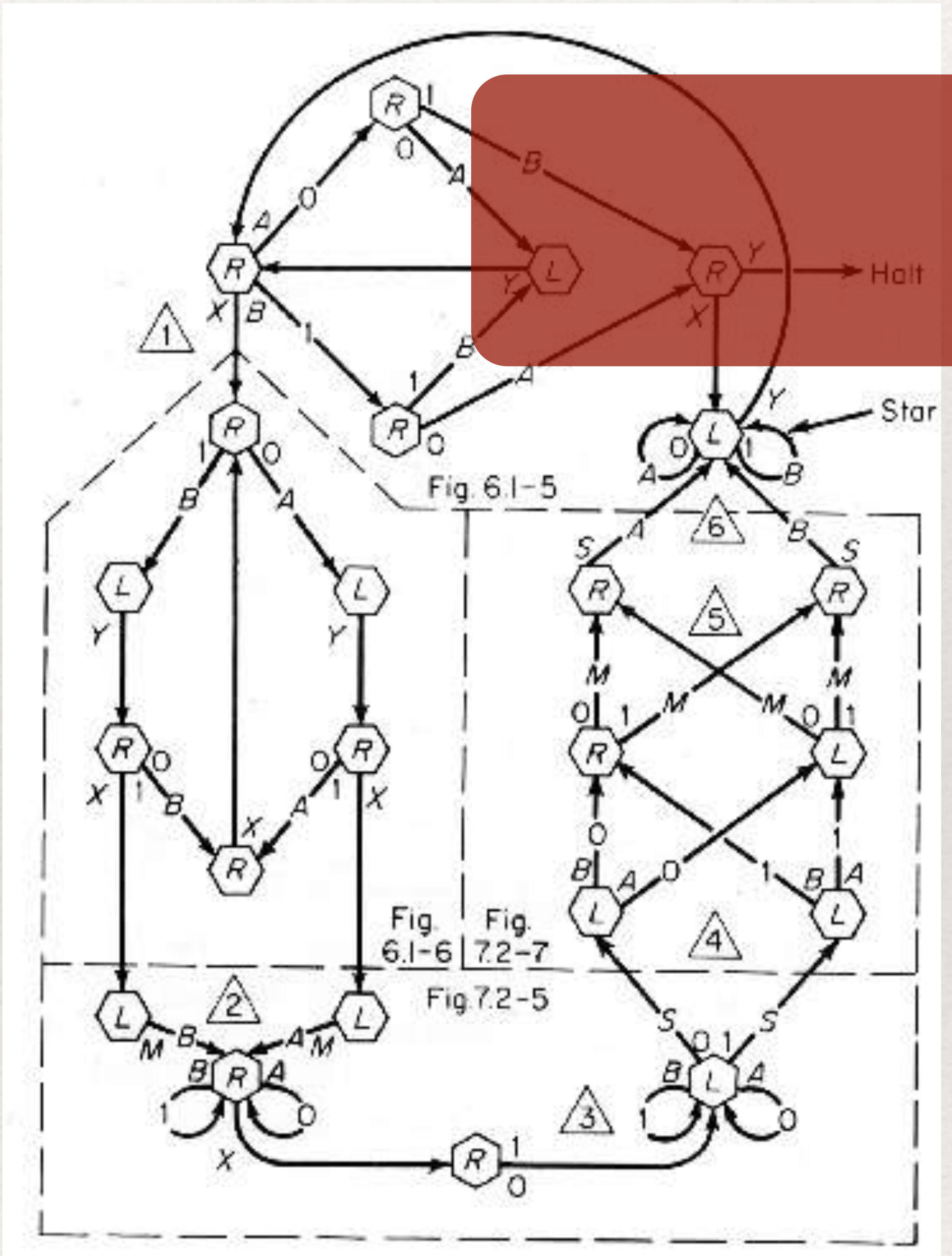


Verification

- In practice, a transition system is a model of some computer system (e.g., a program)
- and B is the set of **bad configurations**, typically where some property of interest is violated. Illustration:

$B =$

B



Well-structured transition systems

(Abdulla,Čerāns,Jonsson&Tsay 2000,
Finkel&Schnoebelen 2001)

- A very interesting class of (infinite) transition systems where **coverability** (a special form of reachability) is **decidable**

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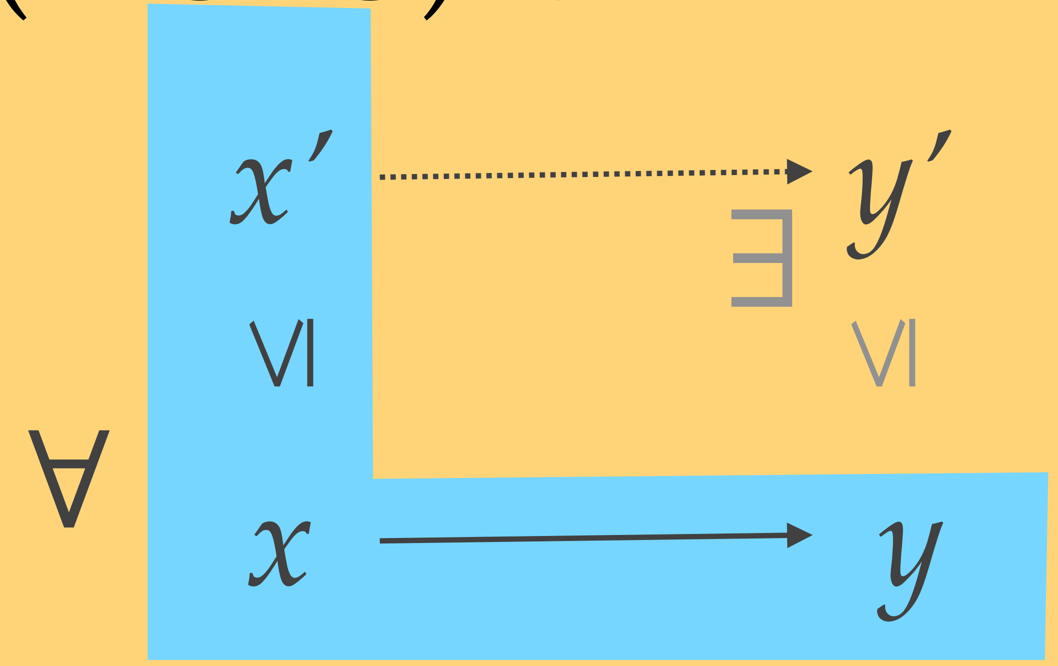

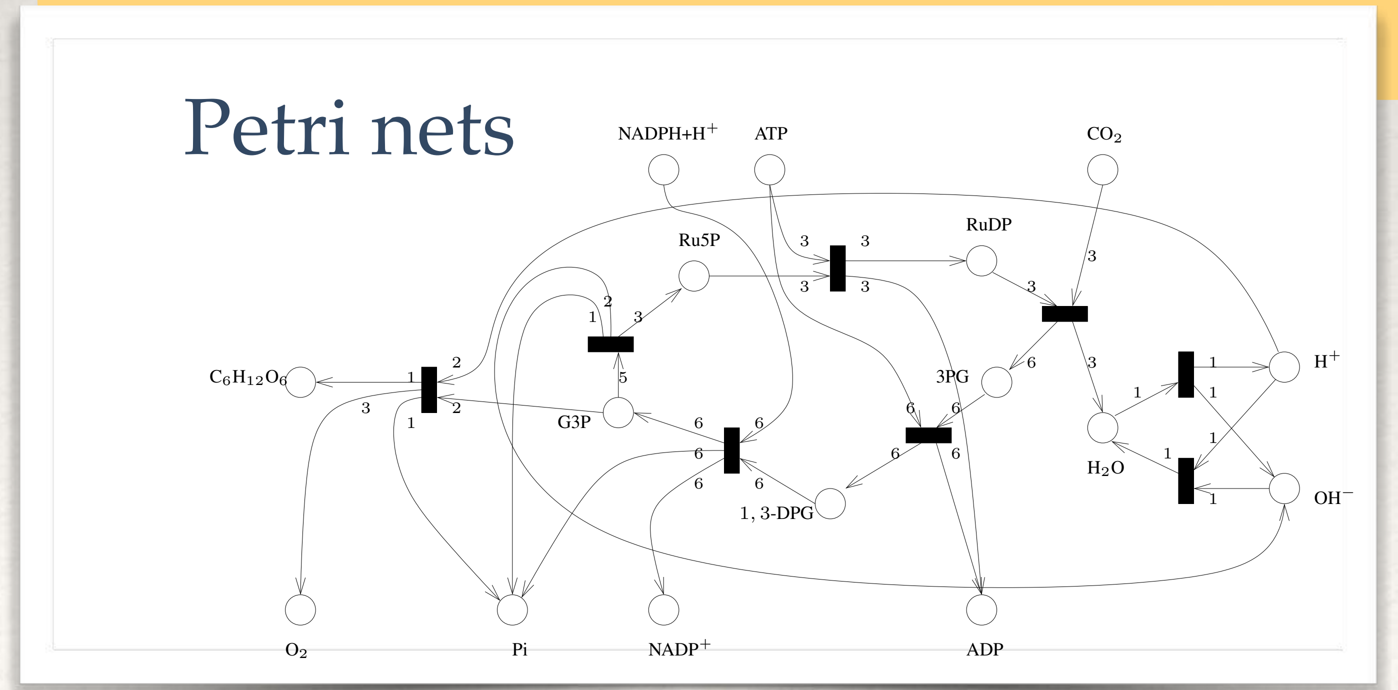

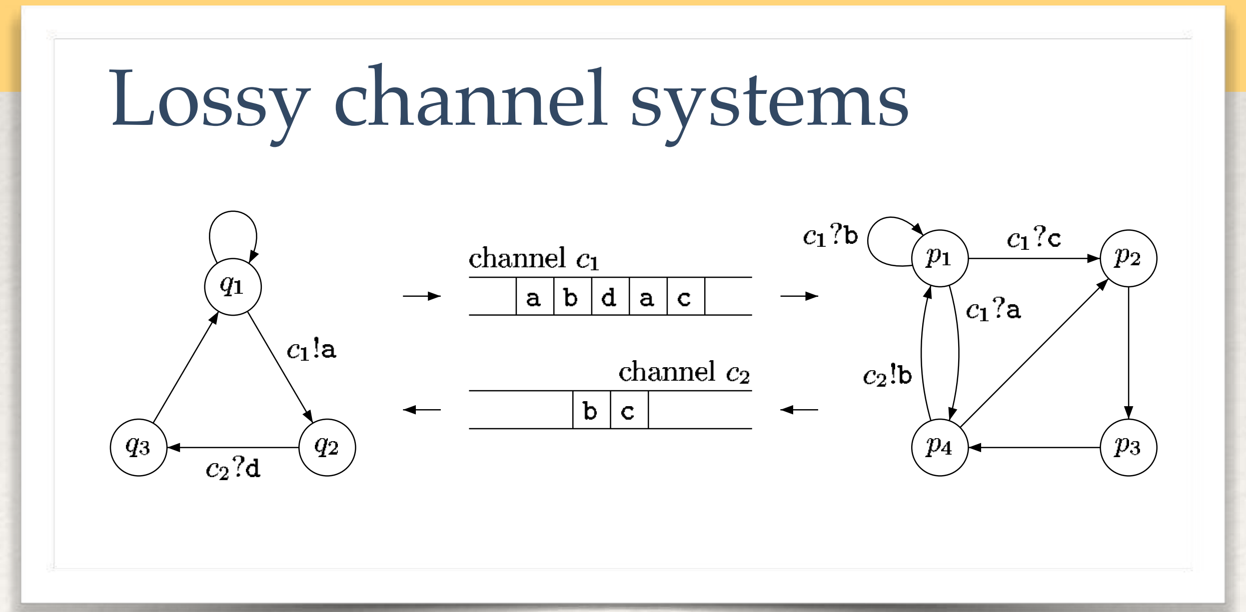

... and many other examples

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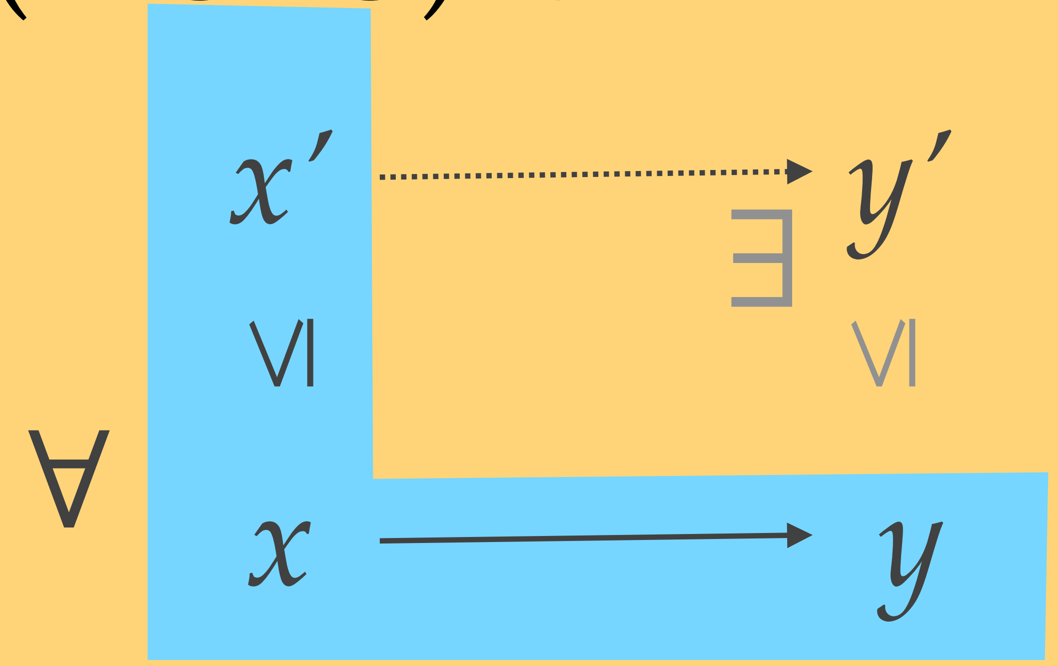

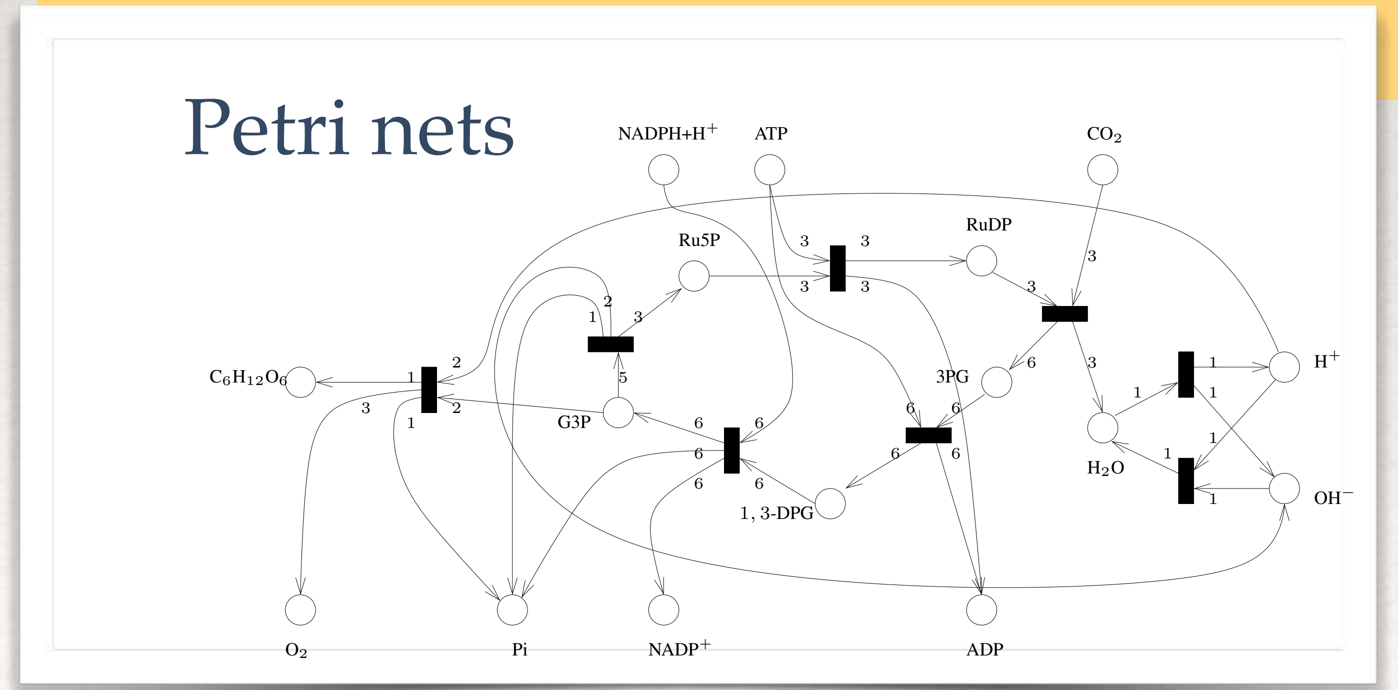

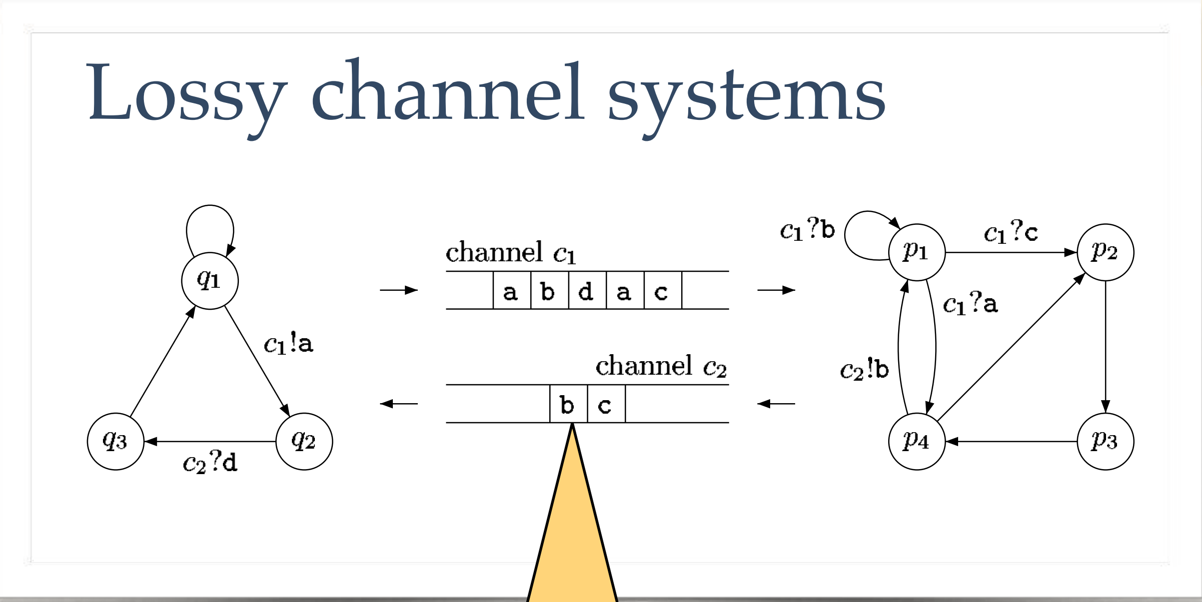

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(needed for decidability… and rather realistic)

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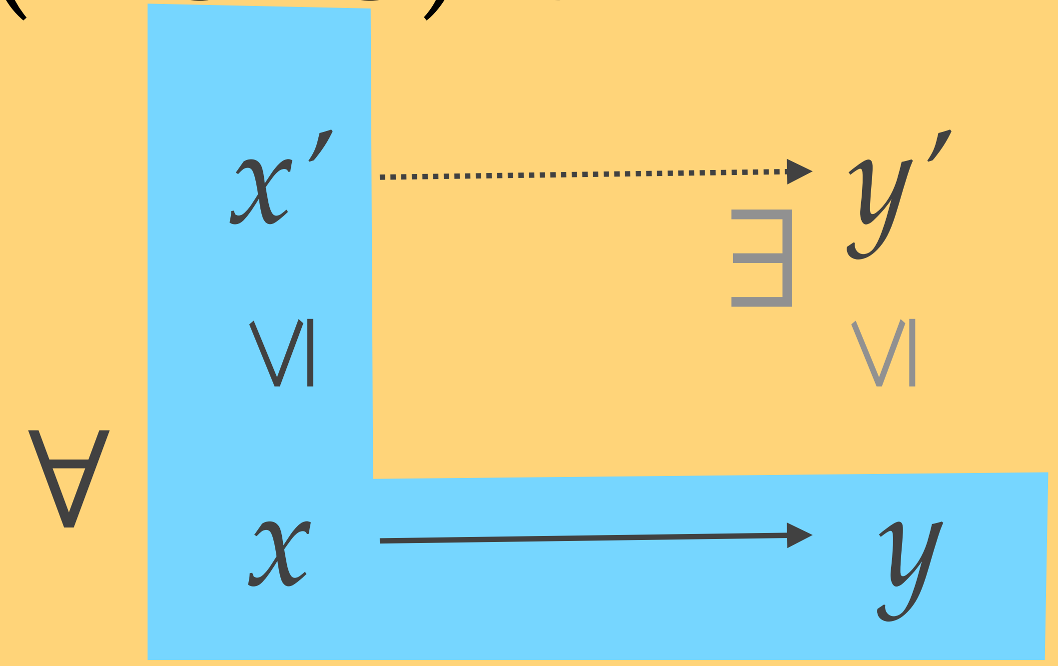

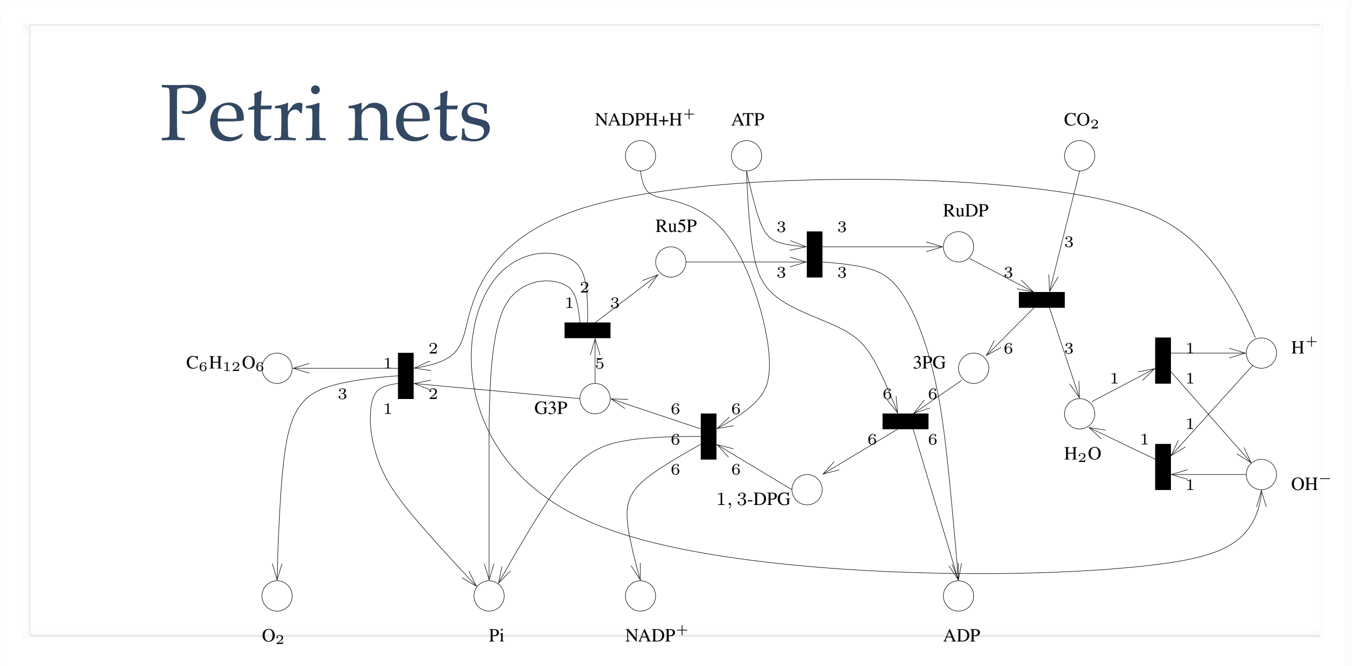

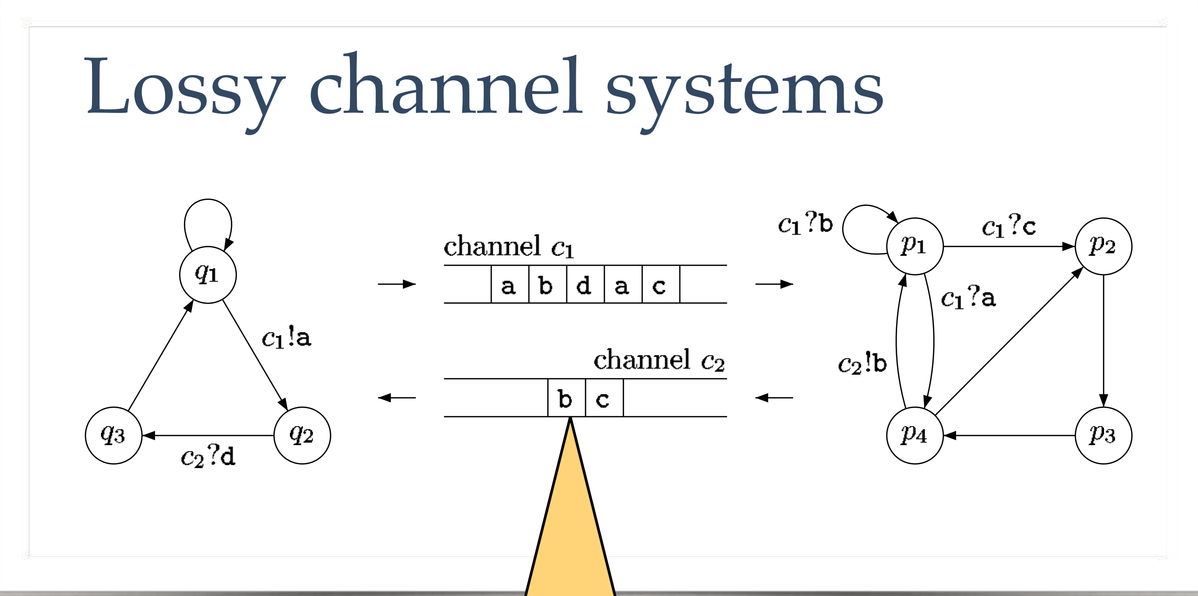

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- In order to understand them, we need…

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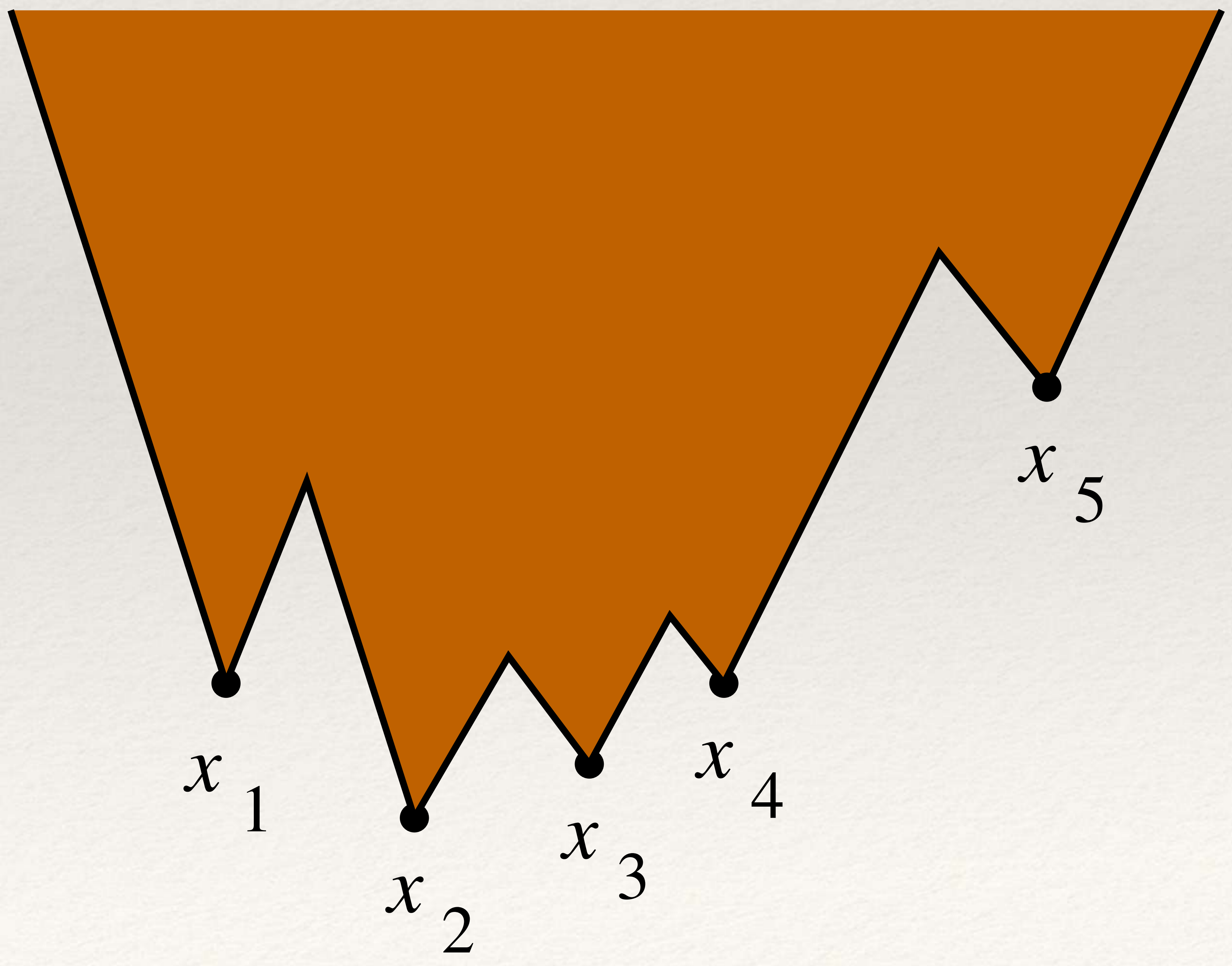

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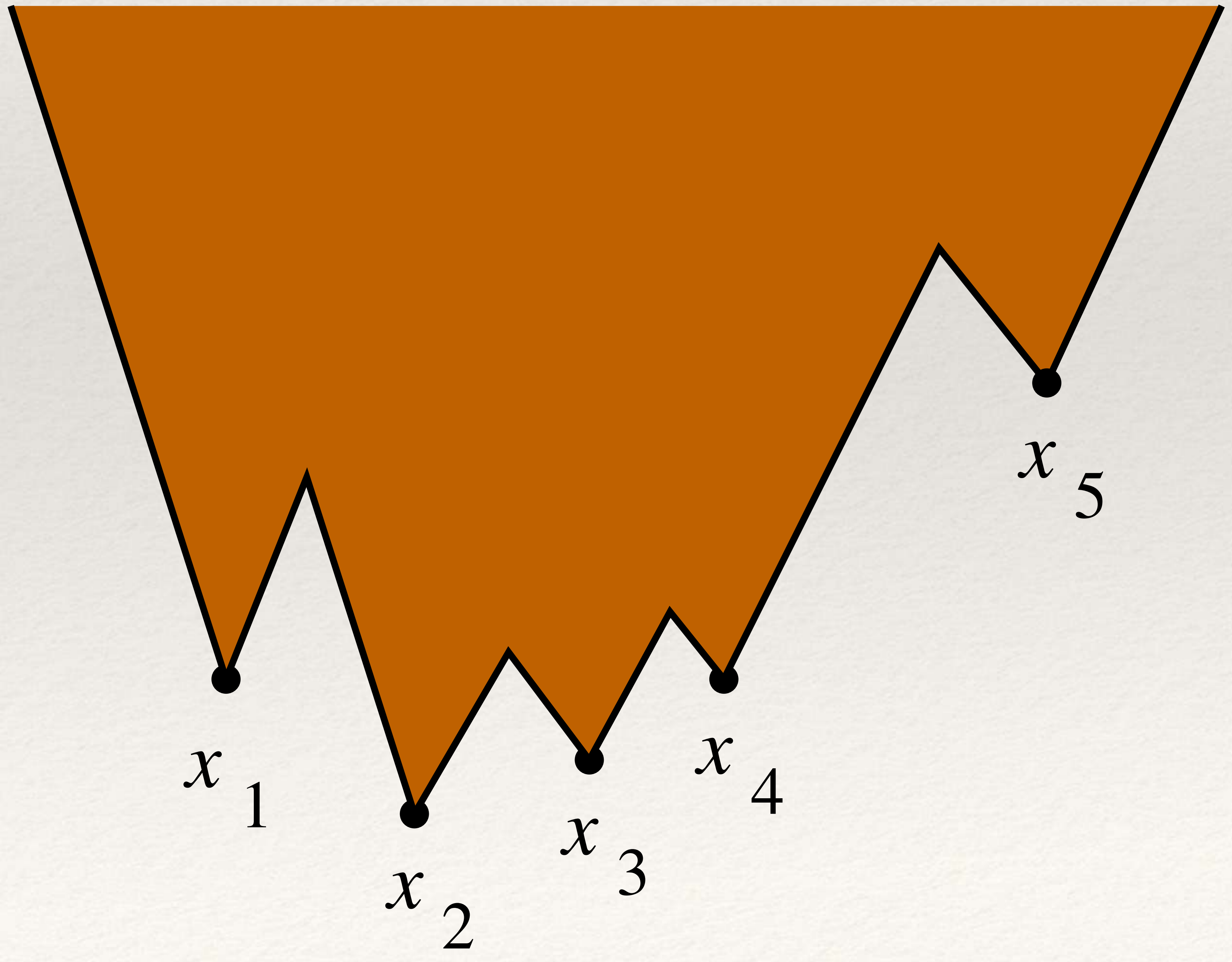
If $x \leq y$ and x is in the set, then so is y



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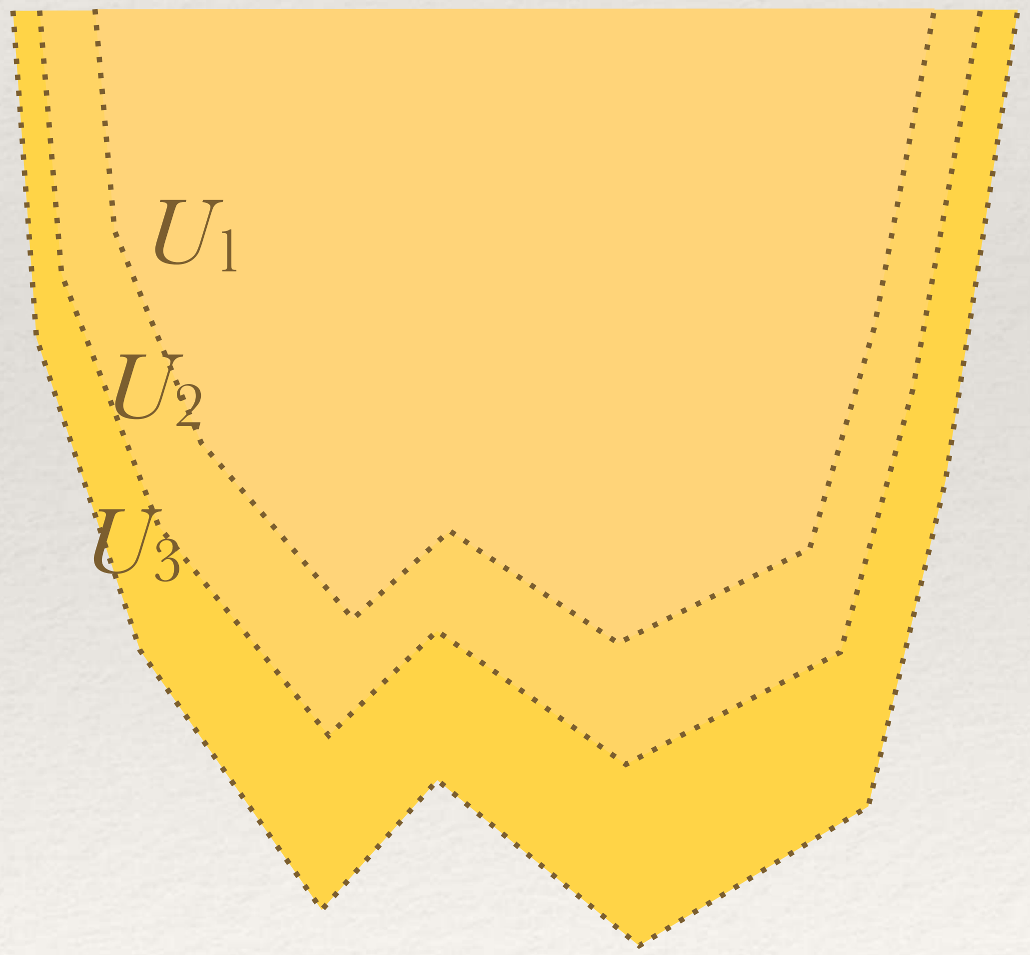
  (5) Every monotonic chain $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots$ of upwards-closed subsets is **stationary**

(i.e., all the sets U_n are equal from some rank on)



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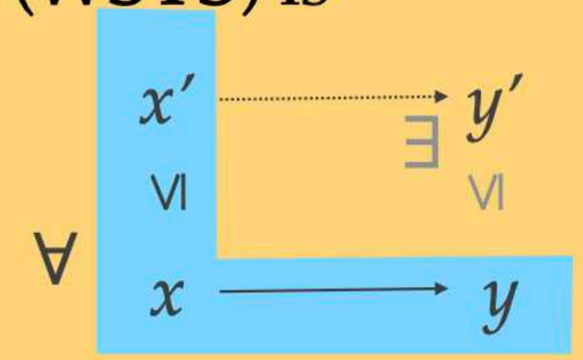

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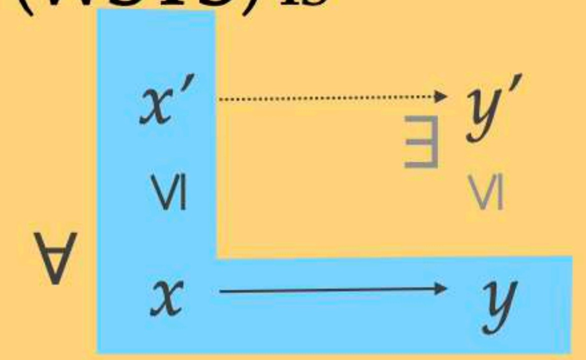

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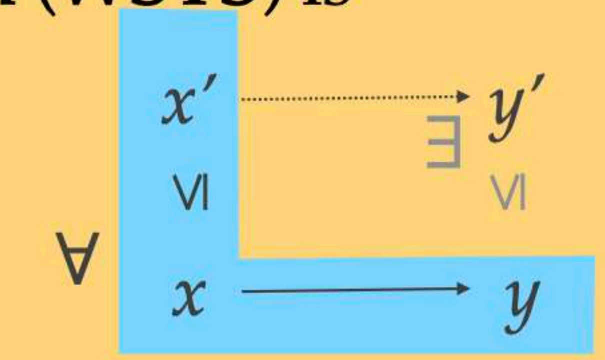
> 
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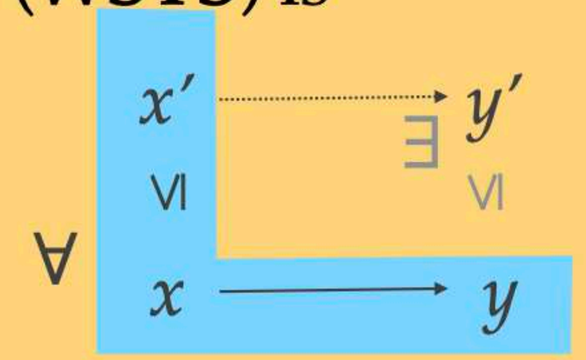

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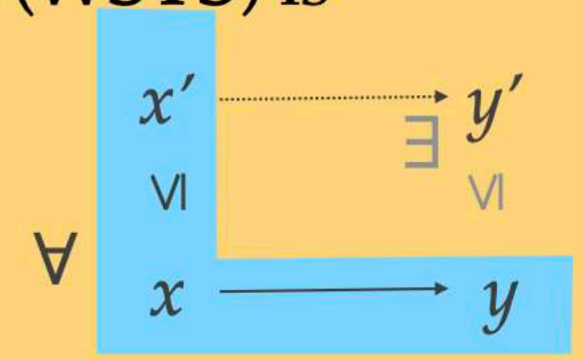

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- Now note that B is reachable from C_0 iff $C_0 \in \mathrm{Pre}^{*}(B)$

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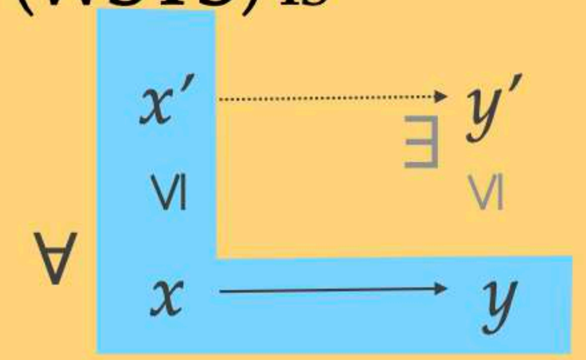

Coverability is decidable

- In order to make this argument precise, we really need to reason with **effective** WSTSs, where
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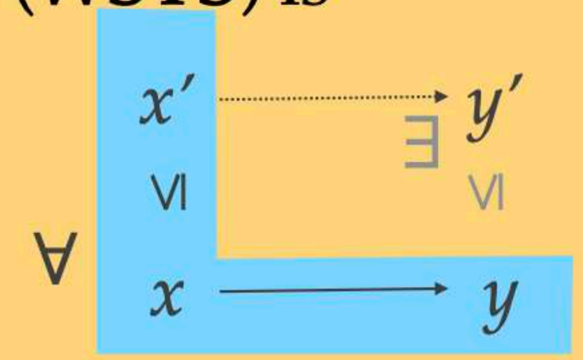

- **Theorem.** (Abdulla et al. 2000, Finkel&Schnoebelen 2001.) Coverability is **decidable** on effective WSTSs.

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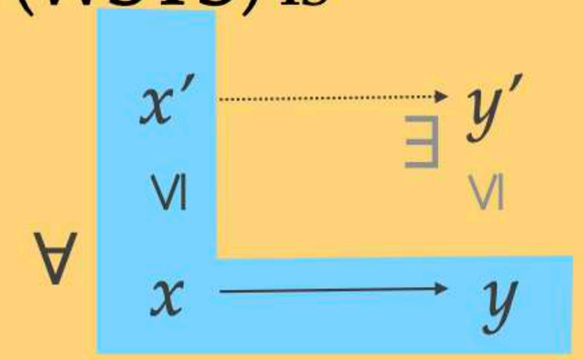

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- Complexity: appalling (**EXPSPACE**-complete for Petri nets, grows faster than Ackermann for lossy channel systems)

Beyond wqos: Noetherian spaces

Going topological

- Every quasi-ordered set $(X, \leq)$ gives rise to a **topological space**, whose open sets are the **upwards-closed** sets (the **Alexandroff topology**)

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- **Proposition.** $(X, \leq)$ is wqo iff X is **Noetherian** in its Alexandroff topology.

- Hence Noetherian spaces generalize wqos

Is the generalization proper?

- **Yes**. Consider $\mathbb{N}_{\mathrm{cof}}$, the set of natural numbers with the **cofinite topology**, whose closed sets are the finite subsets (plus $\mathbb{N}$)

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- **Proposition.** A space X is Noetherian iff
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(Take complements.)

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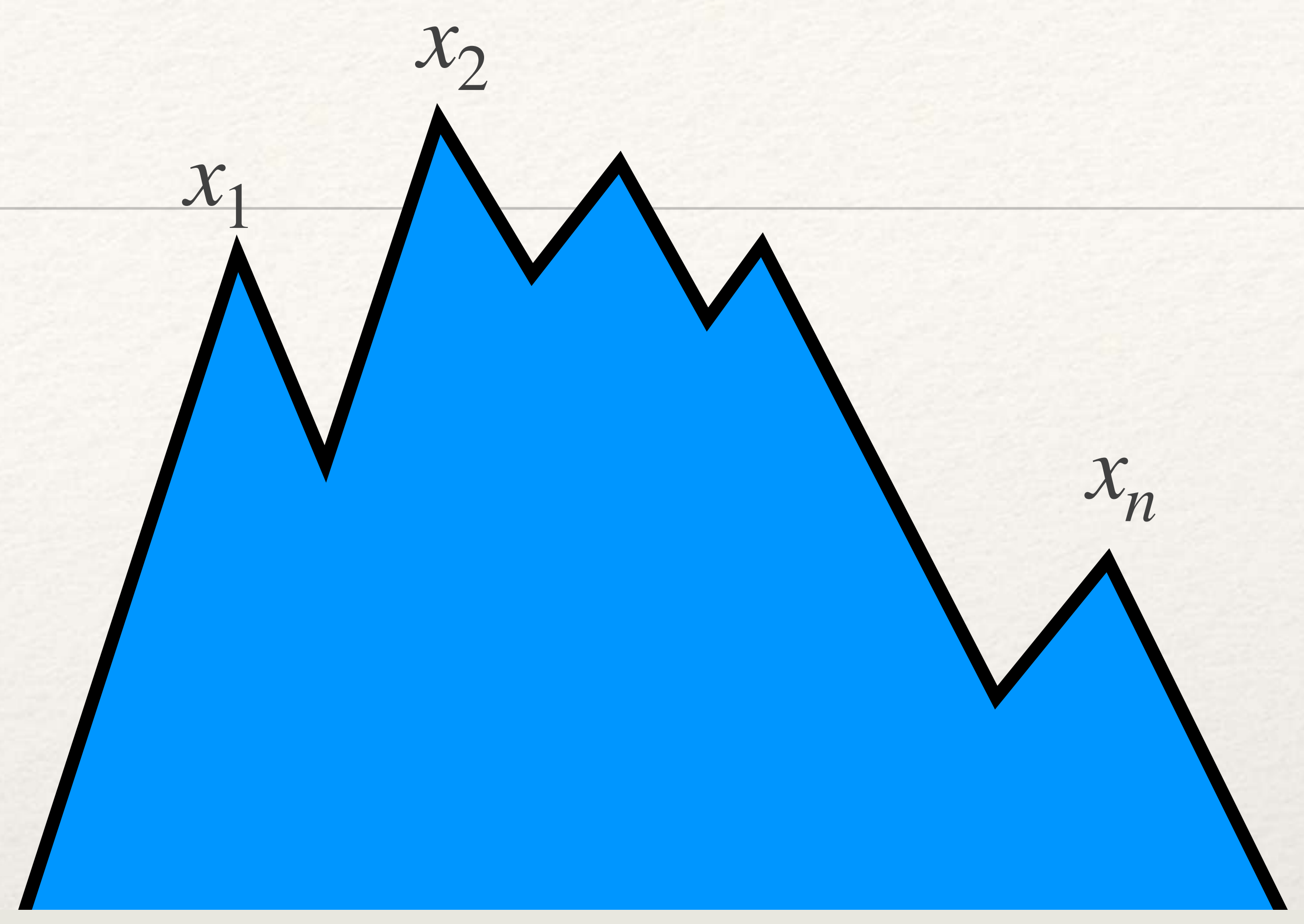
- Oh, wait, why does $\mathbb{N}_{\text{cof}}$ **not** arise from a wqo?

The specialization quasi-ordering

- Every topological space X has a **specialization quasi-ordering**:
  $x \leq y$ iff every open neighborhood of x contains y
  iff x is in the closure of $\{y\}$
- The specialization quasi-ordering of (X in the Alexandroff topology of $\leq$) is $\leq$
- The specialization quasi-ordering of $\mathbb{N}_{\text{cof}}$ is **equality** ($\mathbb{N}_{\text{cof}}$ is T_1)
  and equality is **never** a wqo on an infinite set

So $\mathbb{N}_{\text{cof}}$ is a Noetherian space that does **not** arise from a wqo

Properties T and W



- Let $(X, \leq)$ be a quasi-ordered set.
  Its **finitary** subsets are $\downarrow \{x_1, \cdots, x_n\}$
- The finitary subsets generate the **upper topology**
  It, too, has $\leq$ as specialization quasi-ordering

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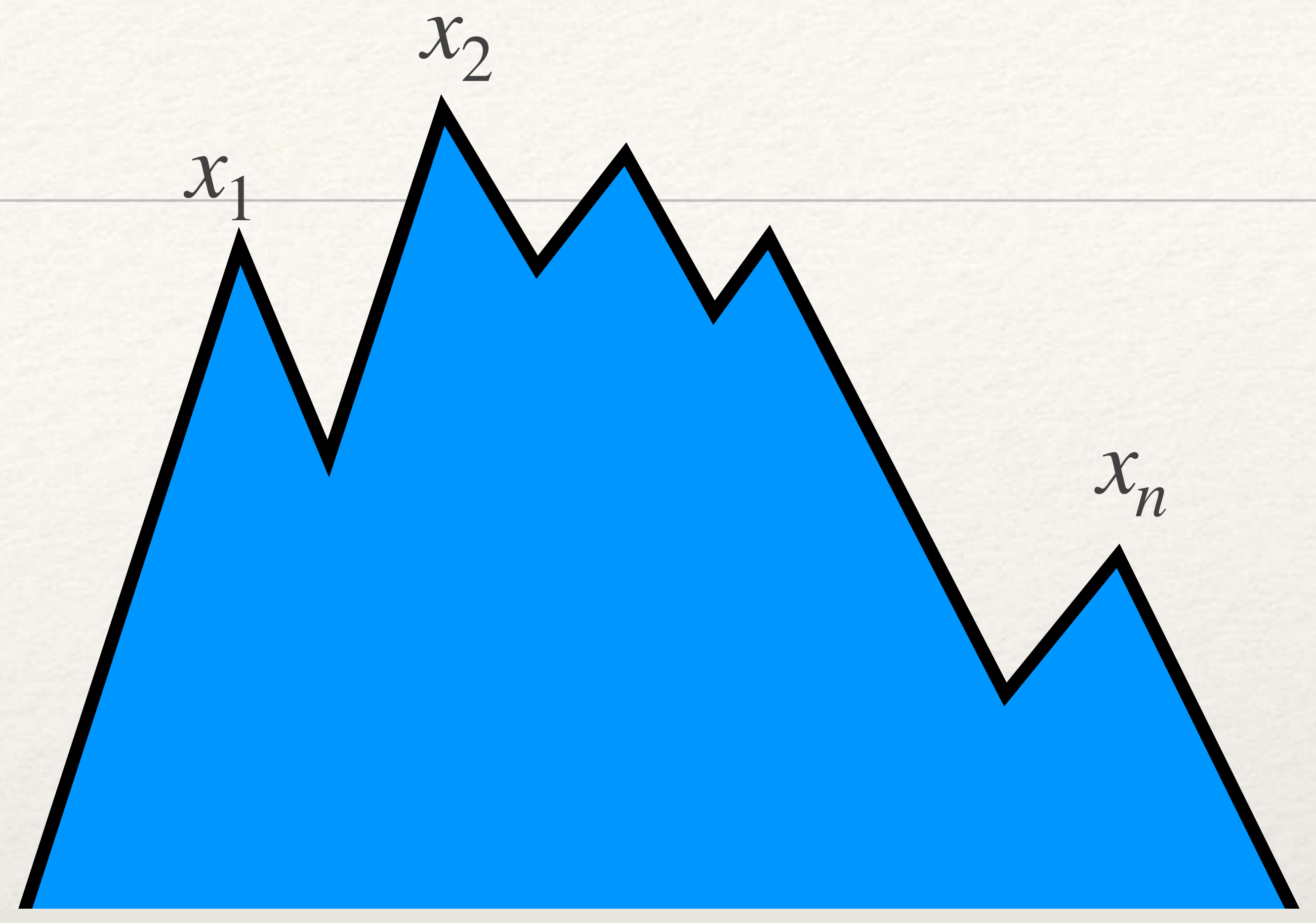

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with $\leq$ as specialization
The Alexandroff topology is the **finest**.

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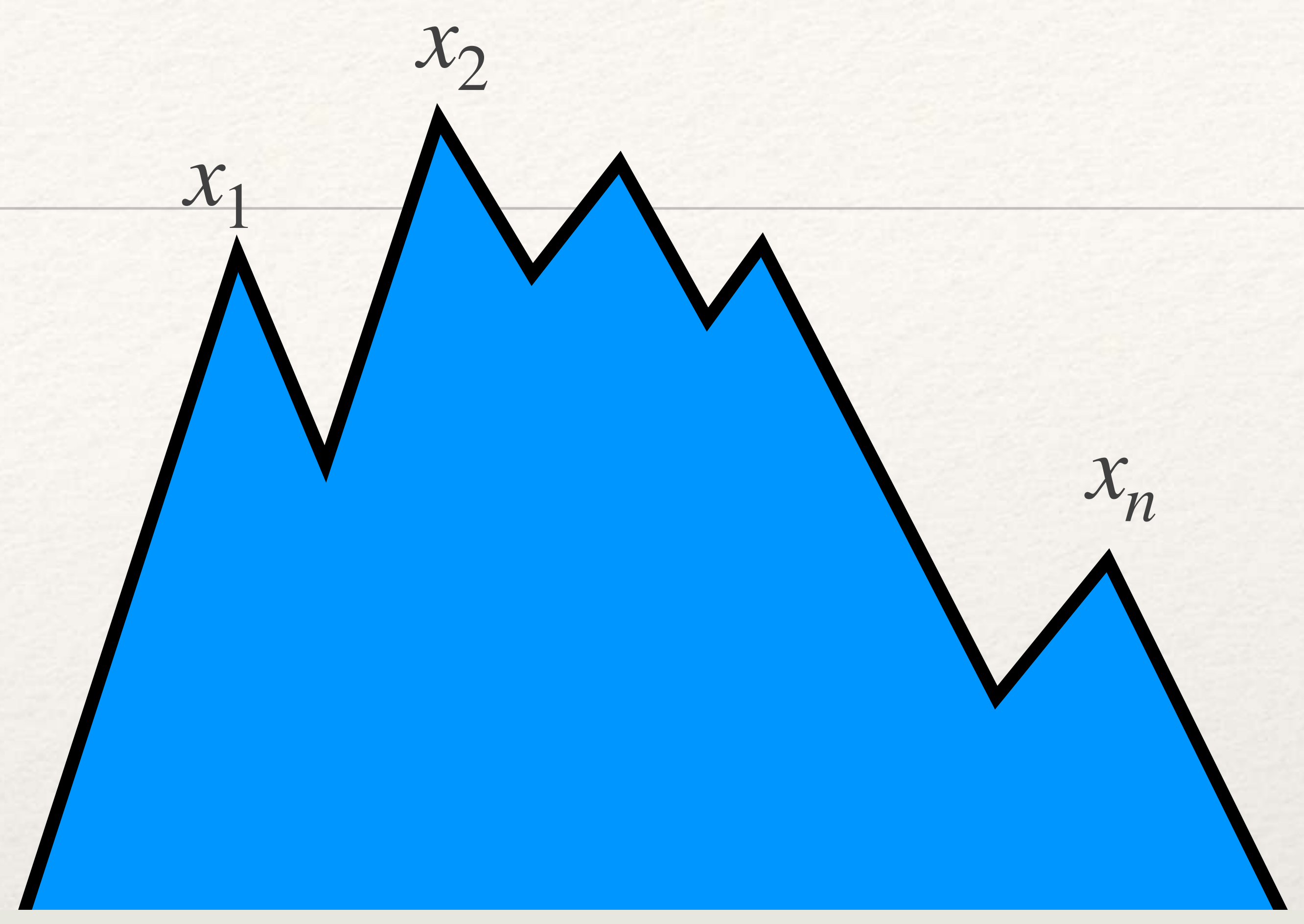

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> The upper topology is the **coarsest** topology with ≤ as specialization
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- **Proposition.** If:
  — X is **well-founded**
  — (Property T) X is **finitary**
  — (Property W) For all $x, y \in X$, $\downarrow x \cap \downarrow y$ is **finitary**
  then X is **Noetherian** in the upper topology
  and the **closed** sets are the **finitary subsets.**

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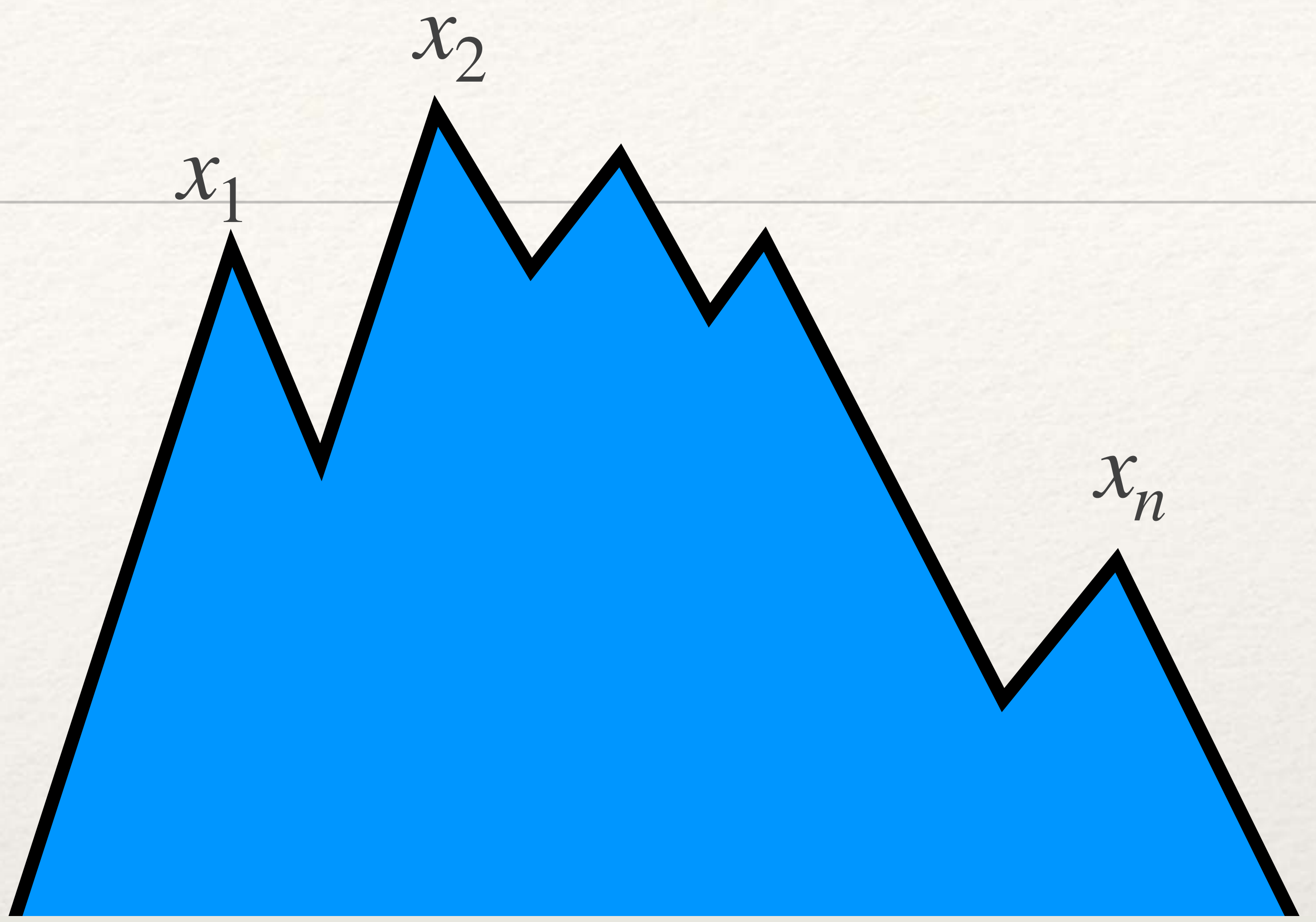

- Let $(X, \leq)$ be a quasi-ordered set.
  Its **finitary** subsets are $\downarrow \{x_1, \cdots, x_n\}$
- The finitary subsets generate the **upper topology**
  It, too, has $\leq$ as specialization quasi-ordering

The upper topology is the **coarsest** topology
with $\leq$ as specialization
The Alexandroff topology is the **finest**.

- **Proposition.** If:
  — X is **well-founded**
  — (Property T) X is **finitary**
  — (Property W) For all $x, y \in X$, $\downarrow x \cap \downarrow y$ is **finitary**
  then X is **Noetherian** in the upper topology
  and the **closed** sets are the **finitary subsets.**

This turns out to be the general form
of all **sober** Noetherian spaces.

The Hoare hyperspace of a Noetherian space

- Every **well-founded inf-semilattice** with top is Noetherian in the upper topology

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(7) $\mathcal{H}X$ is well-founded.

- $\mathcal{H}(X)$ is an inf-semilattice, hence:

Proposition. If X is Noetherian, then $\mathcal{H}X$ is Noetherian.

(That is actually an equivalence.)

The powerset of a Noetherian space

- Equip $\mathbb{P}(X)$ with the **lower Vietoris topology**,

  Subbase of closed sets $\Box C = \{A \in \mathbb{P}(X) \mid A \subseteq C\}$,

  C closed in X

- The lattices of closed sets of $\mathscr{H}X$ and of $\mathbb{P}(X)$ are **isomorphic**, through cl^{-1}: $\downarrow \{C\} \mapsto \Box C$

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Yes, Noetherianness is a **localic** property.
Category of **sober Noetherian** spaces $\cong$
Locales with **no infinite monotonic chain**

The powerset of a Noetherian space

- The **powerset** $\mathbb{P}(X)$ of a Noetherian space X is Noetherian. (with the lower Vietoris topology)

- This came out as a surprise in 2007.
  When X is a wqo (with the Alexandroff topology),
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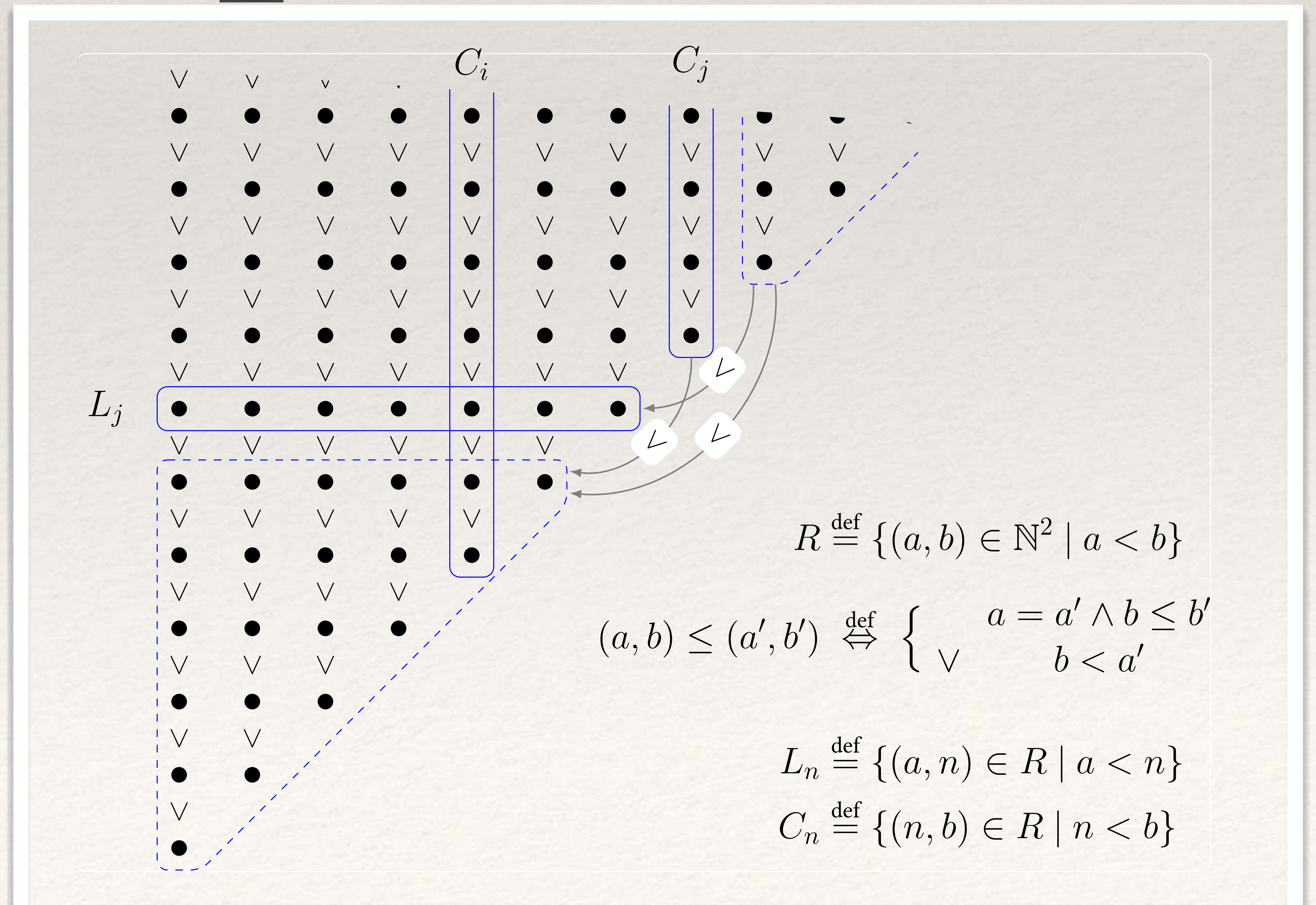

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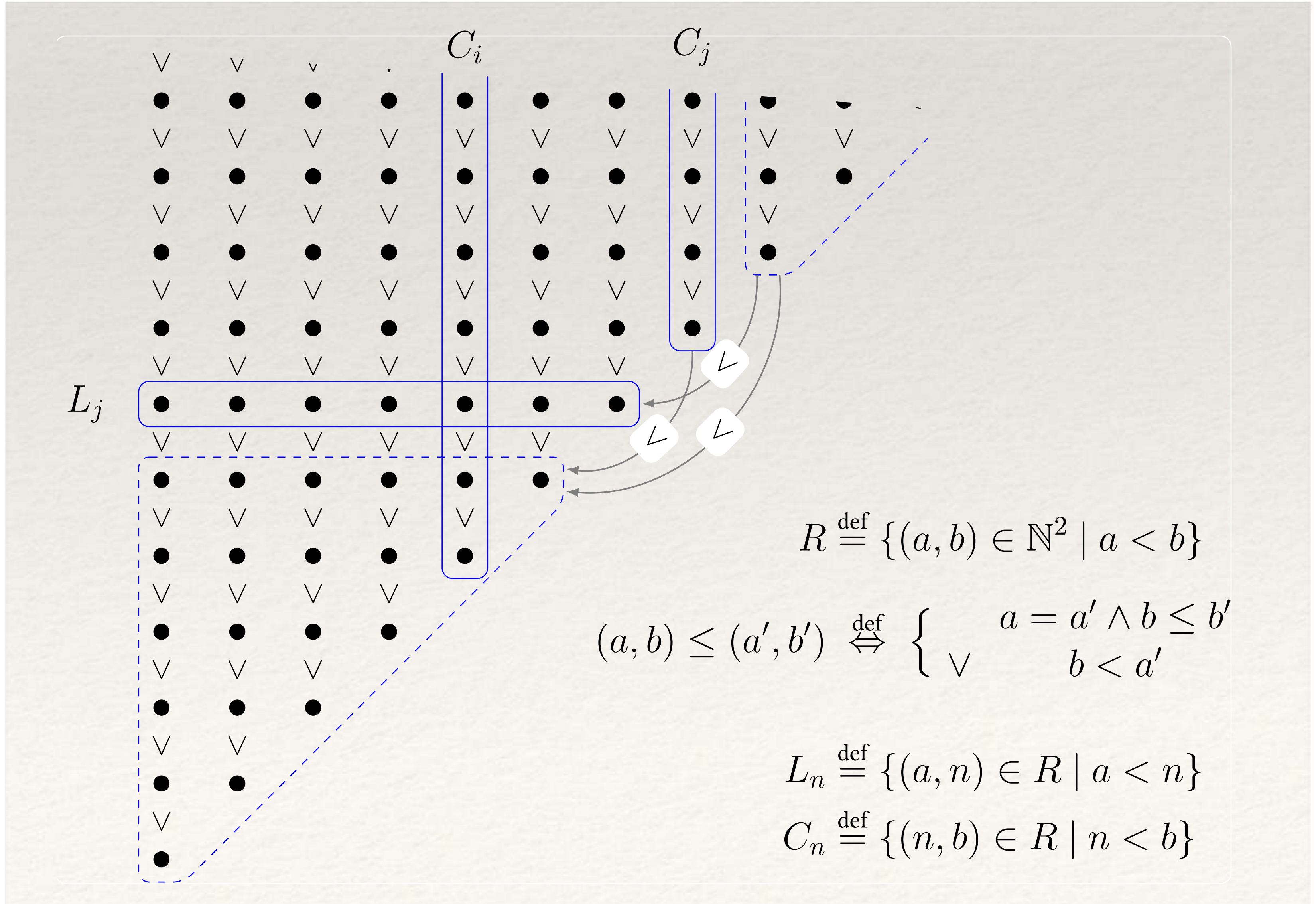
- But $(\mathbb{P}(X), \leq^\flat)$ is **not wqo** for general wqos $(X, \leq)$
  (Rado, 1957)



Finite words

- Let $X^* = \{\text{finite words on } X\}$ with **word topology**:
  basic open sets $\langle U_1, \cdots, U_n \rangle = X^* U_1 X^* \cdots X^* U_n X^*$ (each U_i is open in X)

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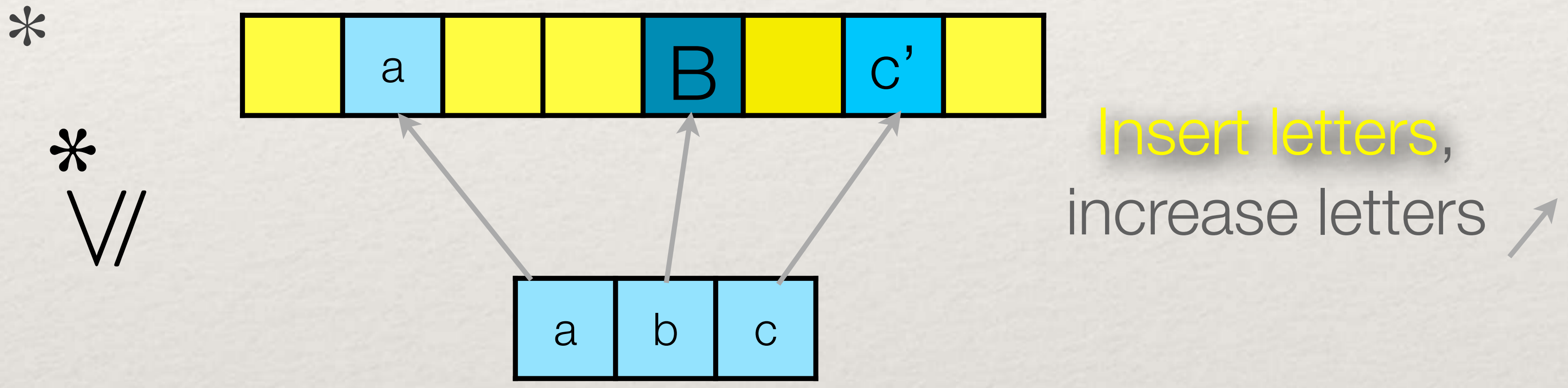

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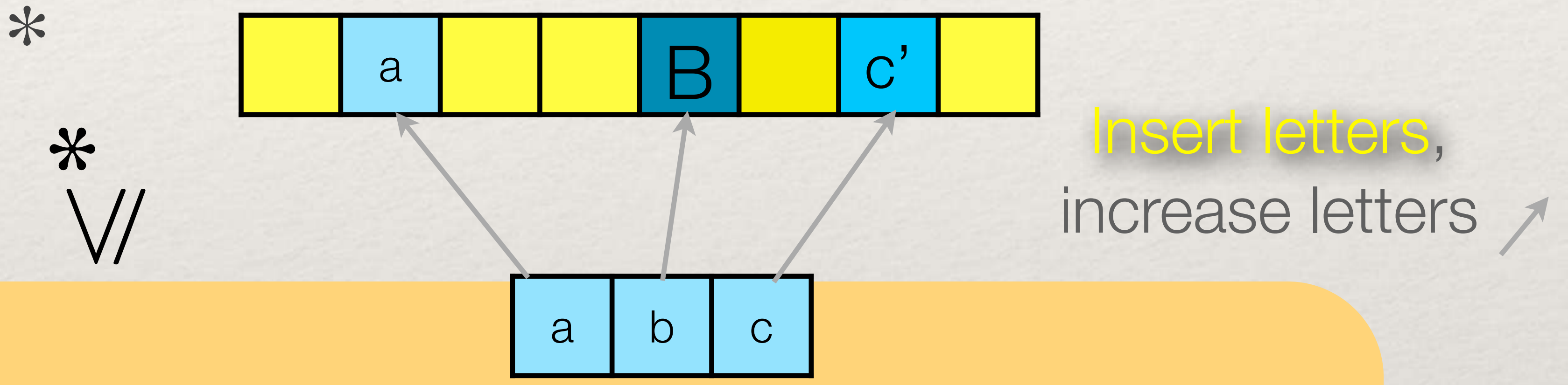
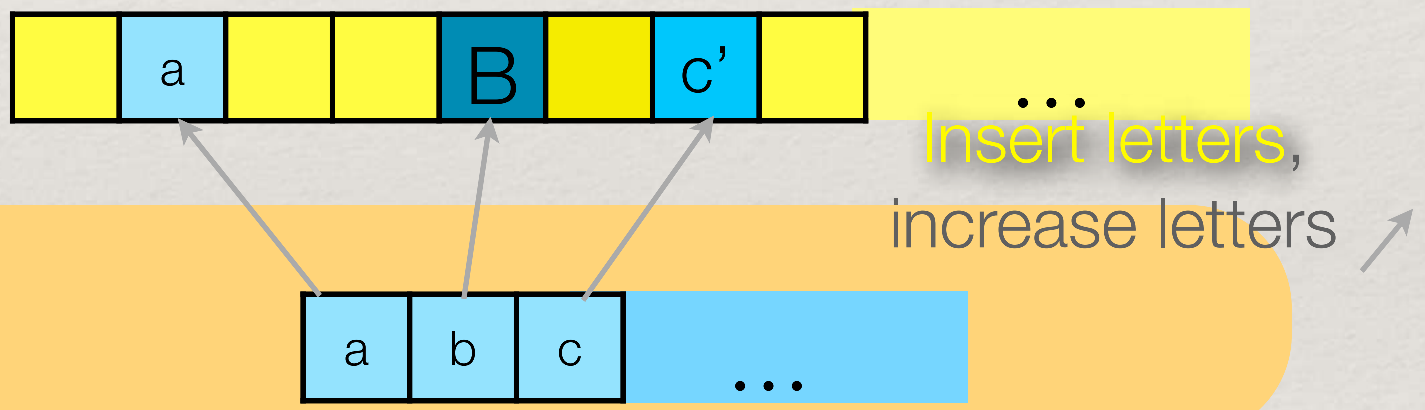


- **Theorem** (JGL 2013). X Noetherian iff X^* Noetherian

  Generalizes Higman's Lemma (Higman 1952): X wqo iff X^* wqo

Infinite words

- Let $X^{\leq\omega}$ = {finite or infinite words on X} with **asymptotic word topology**: subbasic open sets $\langle U_1, \cdots, U_n \rangle = X^* U_1 X^* \cdots X^* U_n X^{\leq\omega}$,
and $\langle U_1, \cdots, U_n; (\infty) V \rangle = X^* U_1 X^* \cdots X^* U_n (X^* V)^\omega$ (U_i, V open in X)
- Specialization quasi-ordering is (infinite) **word embedding**



- **Theorem** (JGL 2021). X Noetherian iff $X^{\leq\omega}$ Noetherian
No equivalent in wqo theory — except if you adopt bqo theory.

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- **Theorem** (JGL, Halfon, Lopez 2022, submitted).
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(Warning: specialization ≠ word embedding in general.)

Topological WSTS

- So Noetherian spaces go beyond wqos,
  but do they have any use?
- Of course they do: a reminder of where they come from
- An application in verification

The origin of Noetherian spaces

- The **spectrum** Spec(R) of a ring R is the set of its **prime ideals** p
- with the **Zariski topology**, whose closed subsets are $\{p \in \mathrm{Spec}(R) \mid I \subseteq p\}$, where I ranges over the ideals of R
- **Fact.** The spectrum of a Noetherian ring (every monotone chain of ideals is stationary) is Noetherian.
- In particular if $R = K[X_1, \cdots, X_n]$ for some Noetherian ring, e.g., $\mathbb{Z}$
- One can **compute** with ideals, represented by **Gröbner bases** (Buchberger 1976)

An application of Gröbner bases in verification

- Verification of **polynomial programs** (Müller-Olm&Seidl 2002)
- Propagates ideals of $\mathbb{Z}[X_1, \cdots, X_n]$ **backwards**, as in the Pre* algorithm
  ($X_1, \ldots, X_n$ = variables of the program)
- Terminates because every monotonic chain $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$ of ideals is **stationary**
- … very similar to Pre* on WSTS, but the (infinite) transition system underlying a polynomial program is **not** a WSTS (inclusion between ideals **not** a wqo)

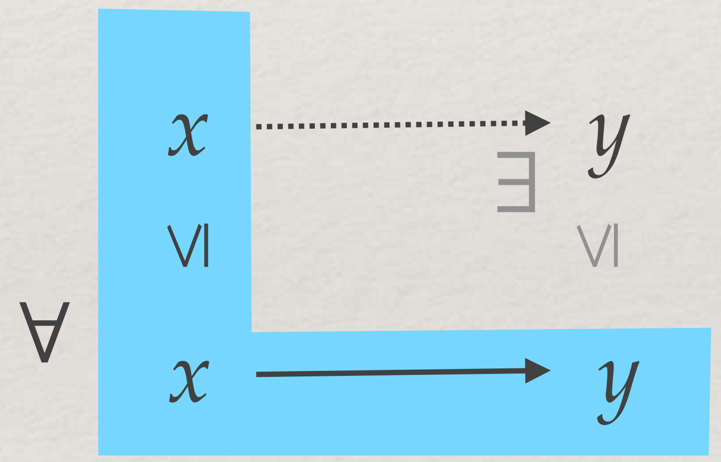
```
while (*) {
  if (*) { x=2; y=3; }
    else { x=3; y=2; }
  x = x*y-6; y=0;
  if (x²-3*x*y==0)
     while (*) { x=x+1; y=y-1; };
  x = x²+x*y;
}
```

Topological WSTS

(JGL 2011)

- **Definition.** A **topological WSTS** is a transition system $(X, \rightarrow)$ with a **Noetherian topology** $\leq$ on X satisfying **lower semicontinuity**:
  for every open subset U, $\mathrm{Pre}(U)$ is open

- Namely, replace **wqo** by **Noetherian**
  monotonicity by **lower semicontinuity**



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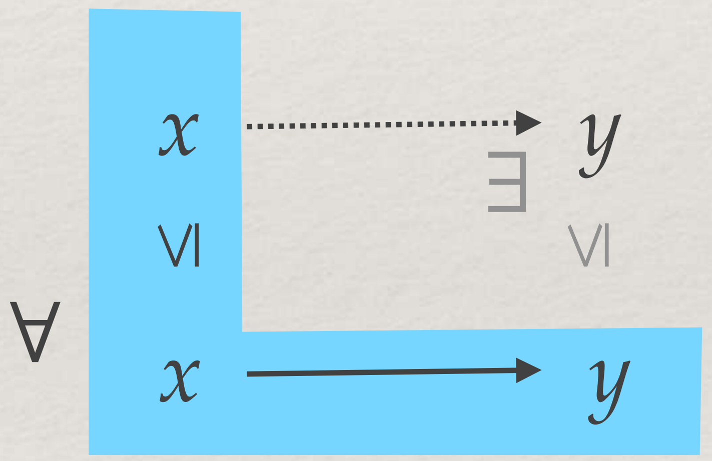

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- **Polynomial programs** are topological WSTS
  — in the Zariski topology of $\mathrm{Spec}(\mathbb{Z}[X_1, \cdots, X_n])$

Topological coverability is decidable

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  INPUT: an initial configuration x_0,
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  — open sets are representable
  — $\subseteq$ is decidable on open sets
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- **Theorem** (JGL 2011.) Topological coverability is **decidable** on effective topological WSTSs.

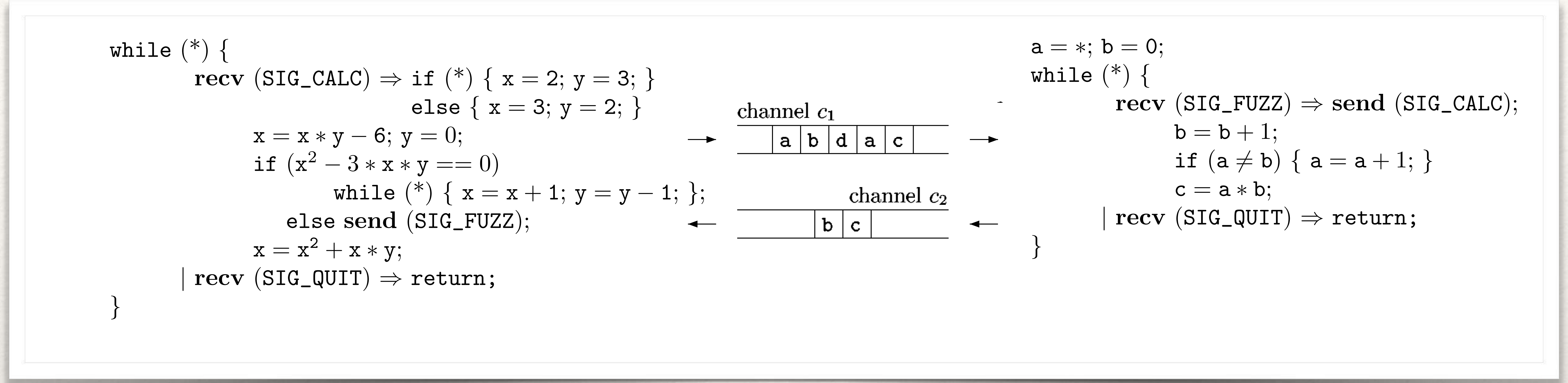
```
fun pre* U =
    let V = pre U
    in
       if V⊆U
          then U
       else pre* (U ∪ V)
    end;

fun coverability (s, B) =
    s in pre* (B);
```

- The algorithm is the same as with WSTSs.

Concurrent polynomial programs

(JGL 2011)

- Finite networks of polynomial programs $P_1, \ldots, P_m$ communicating through lossy communication queues on a finite alphabet Σ



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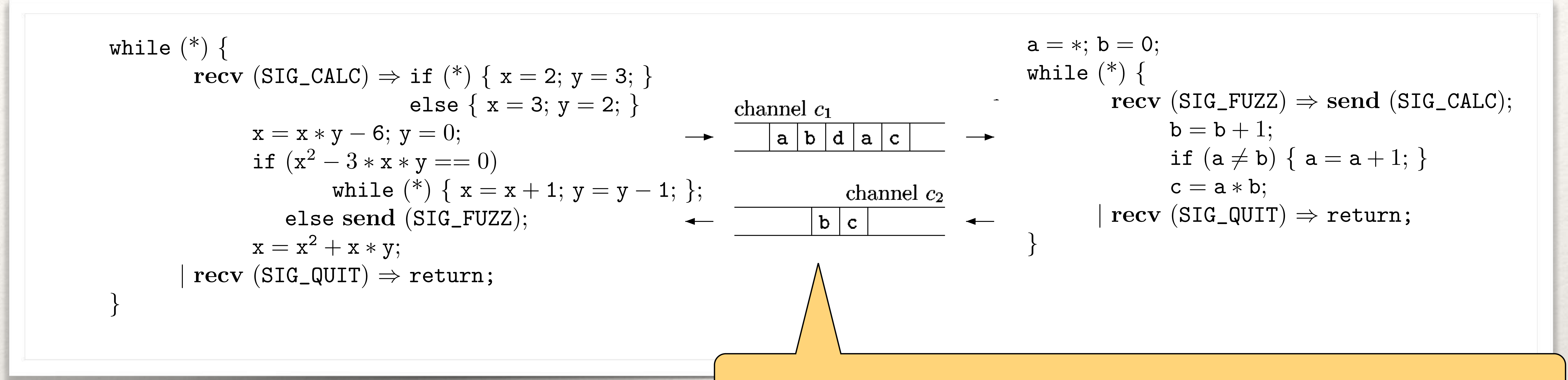

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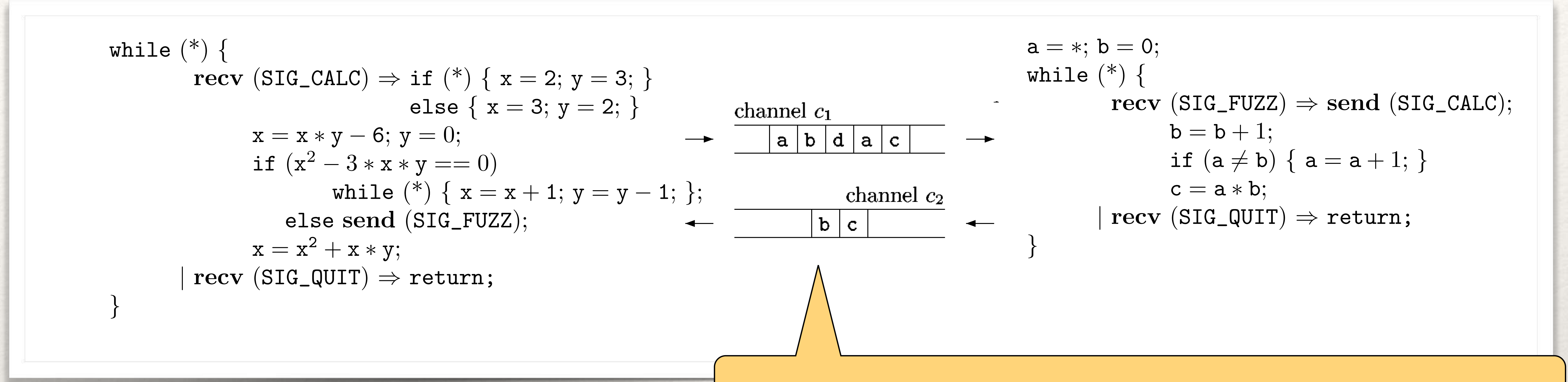

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- State space = finite **product** of
  — **spectra** of polynomial rings $\mathbb{Z}[X_1, \cdots, X_n]$, one for each P_i
  — Σ^*, with **word topology**, one for each communication queue

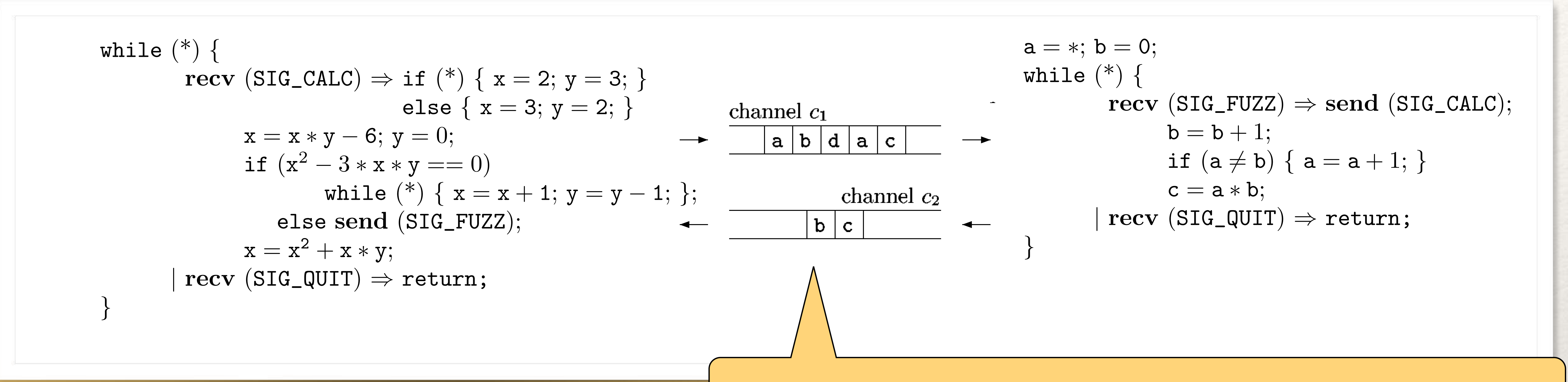
  This is **Noetherian**, because:

- **Proposition.** Any finite product of Noetherian spaces is Noetherian.

Concurrent polynomial programs

(JGL 2011)

- Those are **topological WSTSs** (lossiness necessary) Hence:



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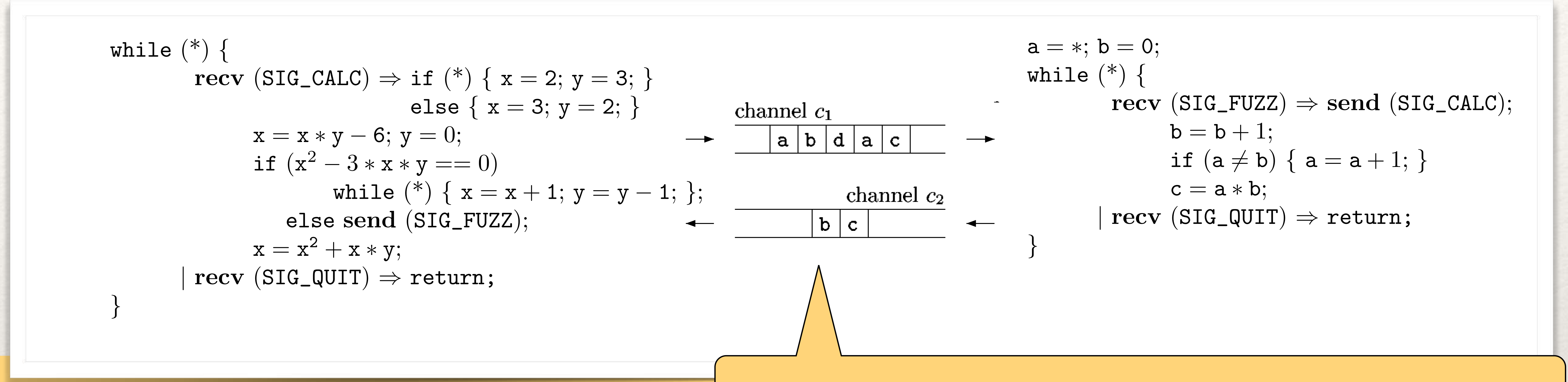

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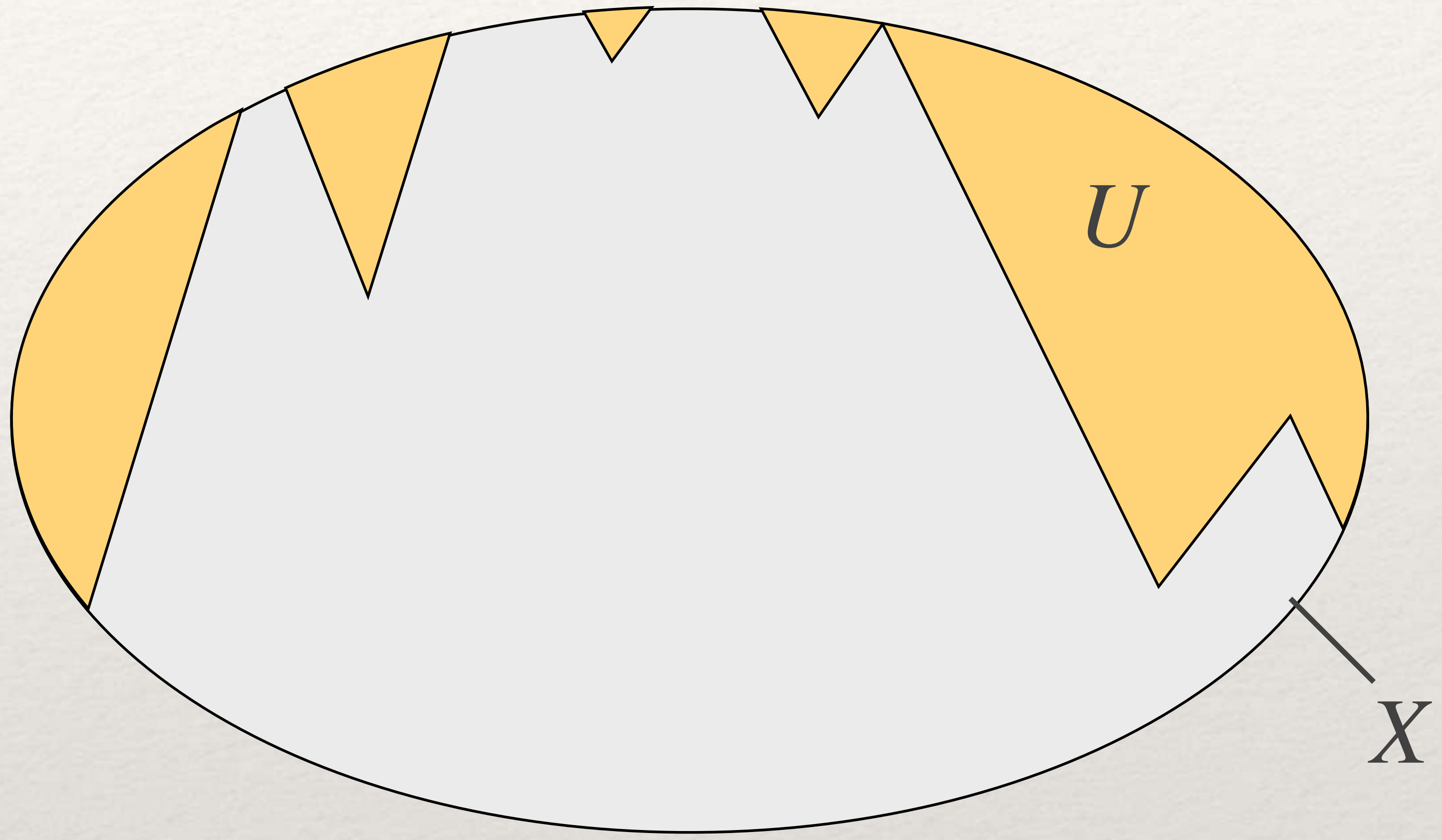
- **Theorem** (JGL 2011).
  Topological coverability is **decidable** for concurrent polynomial programs.

- You still have to prove effectivity.
  For that, you need to find a representation for open sets.
  But open sets are **no longer** of the form $\uparrow \{x_1, \cdots, x_n\}$

Representations, sobrifications

Representing open sets: the trick

- Embed state space X into its **sobrification** X^s



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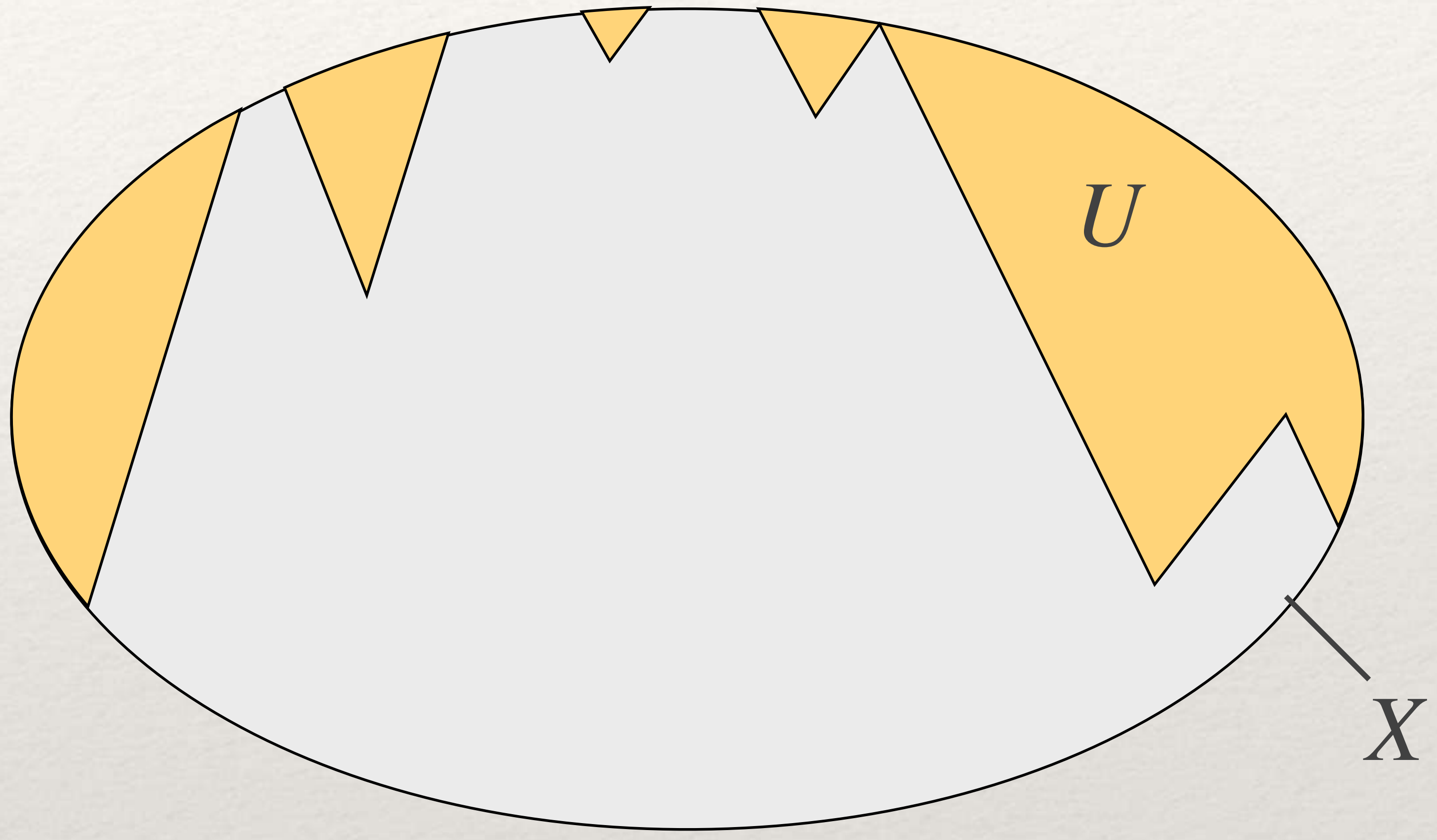
Oops, I have not said what that was, have I?



Sober spaces and sobrifications

- A closed set $F \in \mathscr{H}X$ is **irreducible** iff
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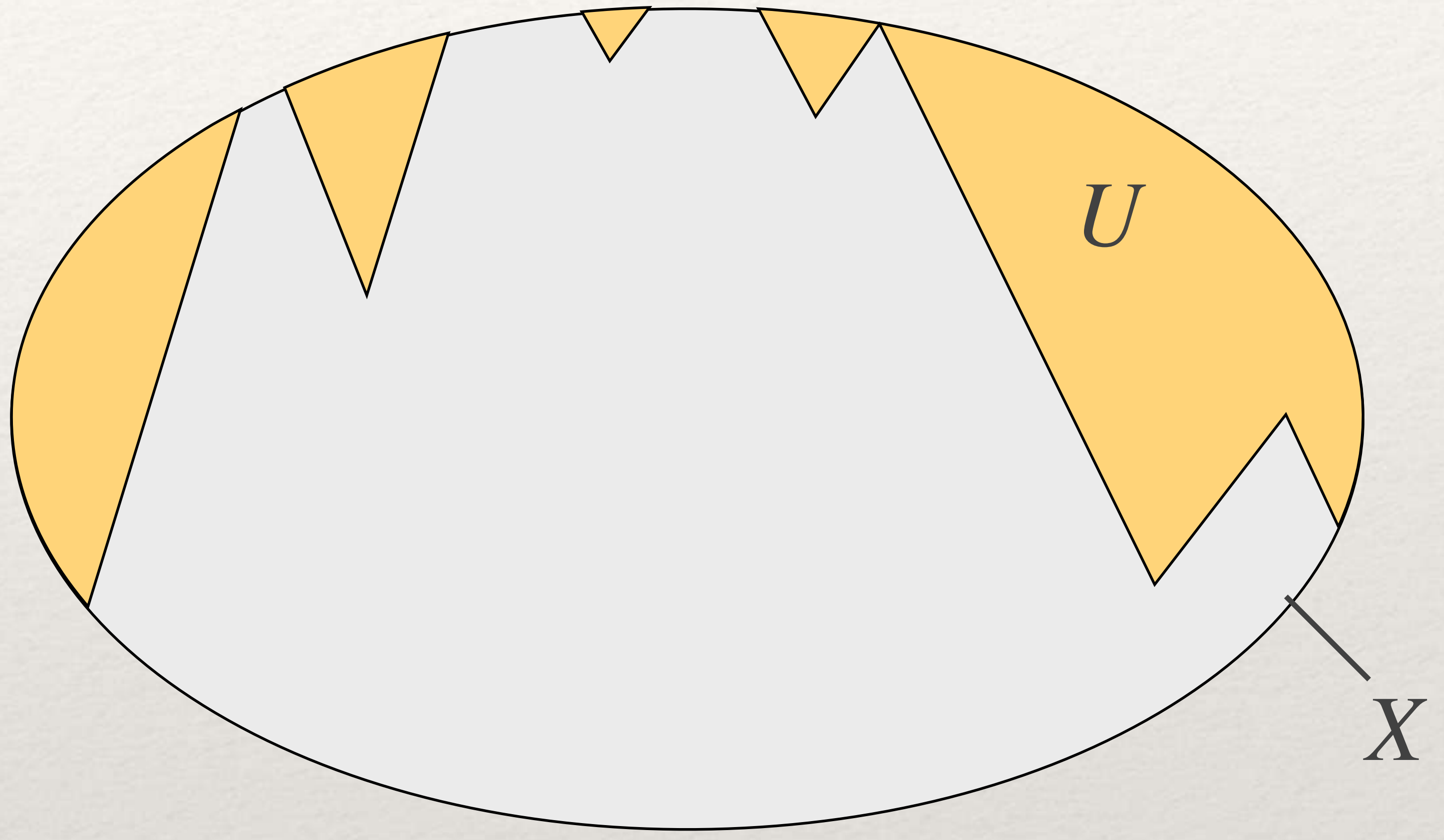

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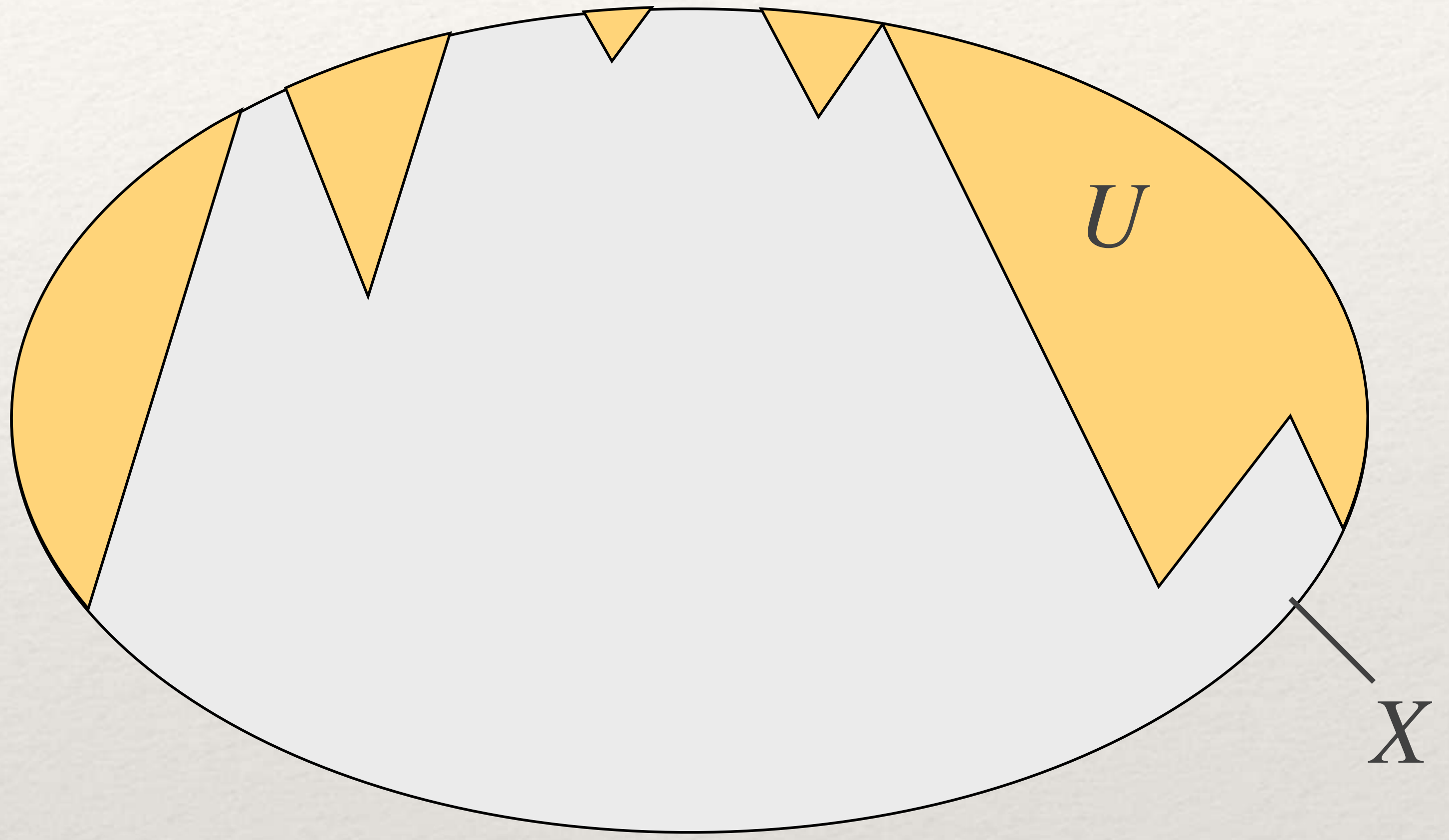

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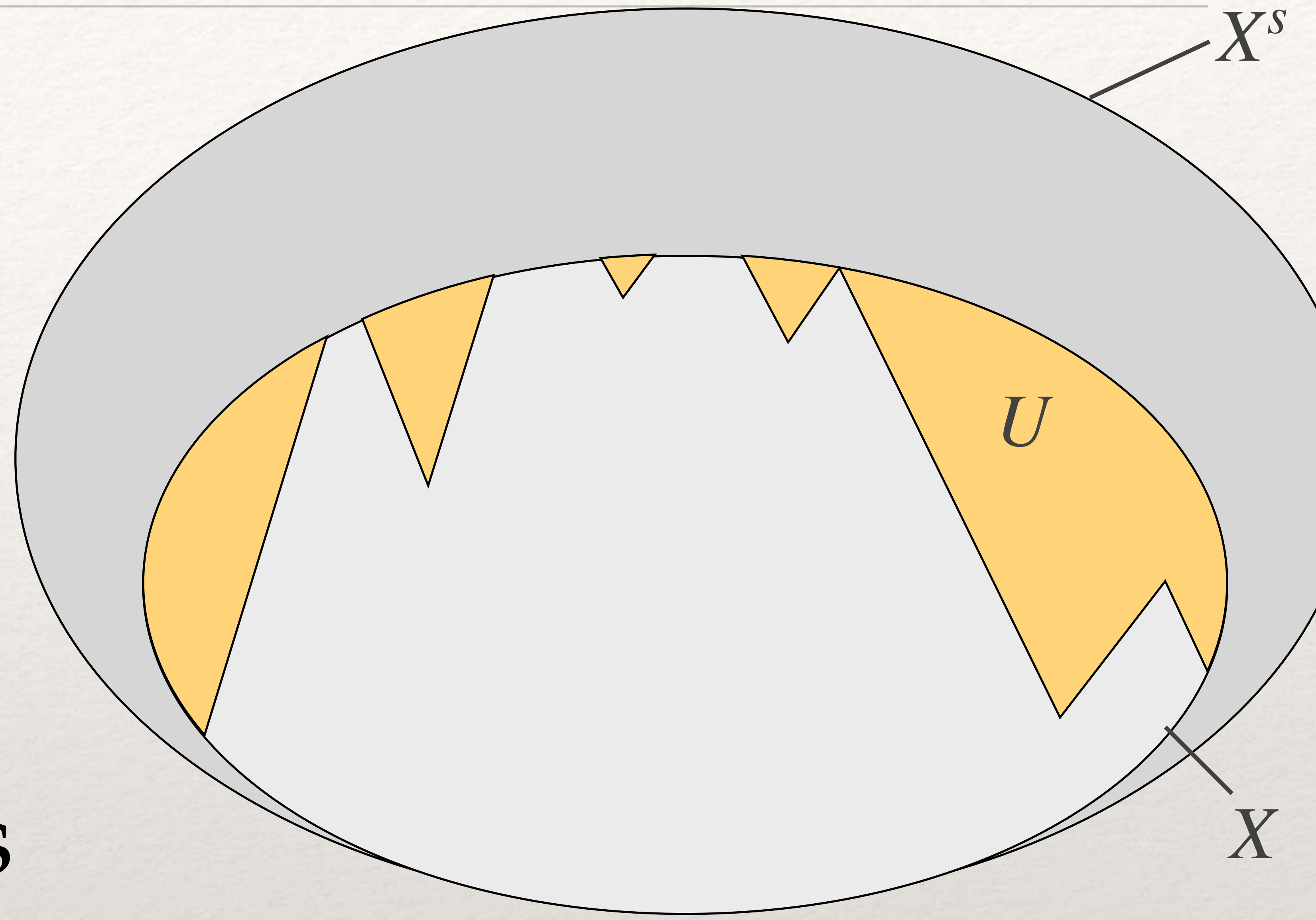

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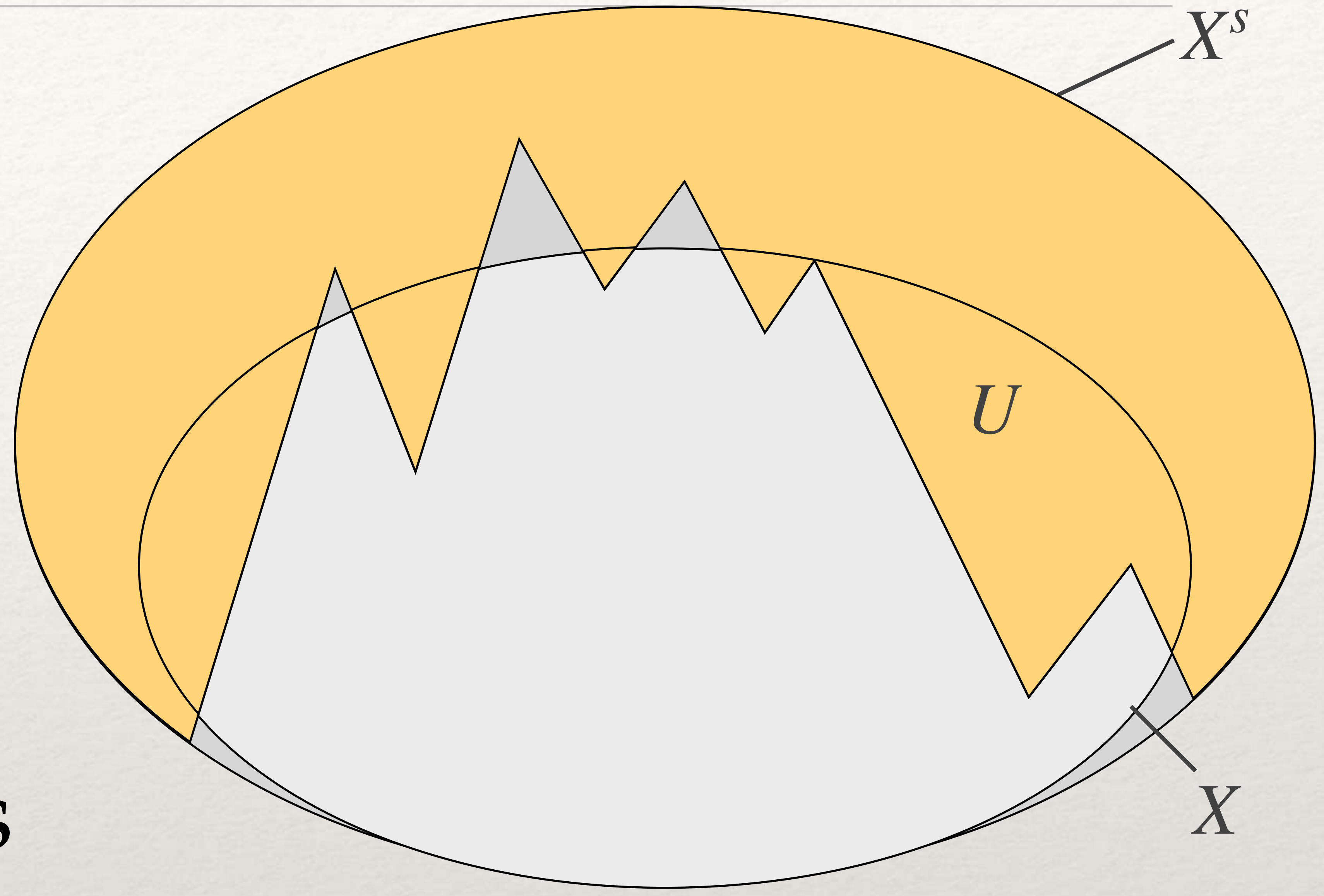

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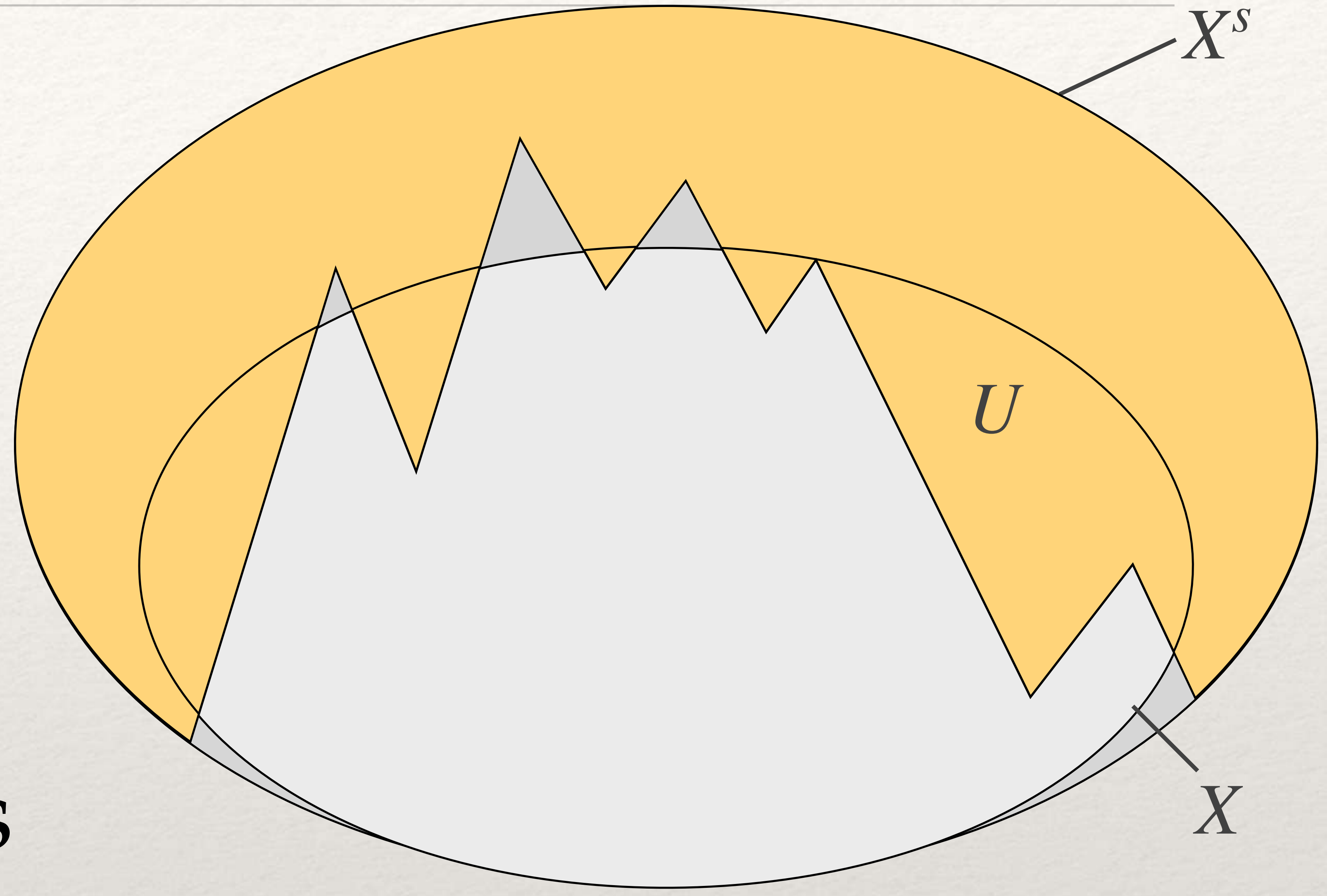
- X and X^s have isomorphic lattices of open subsets

Sober spaces and sobrifications



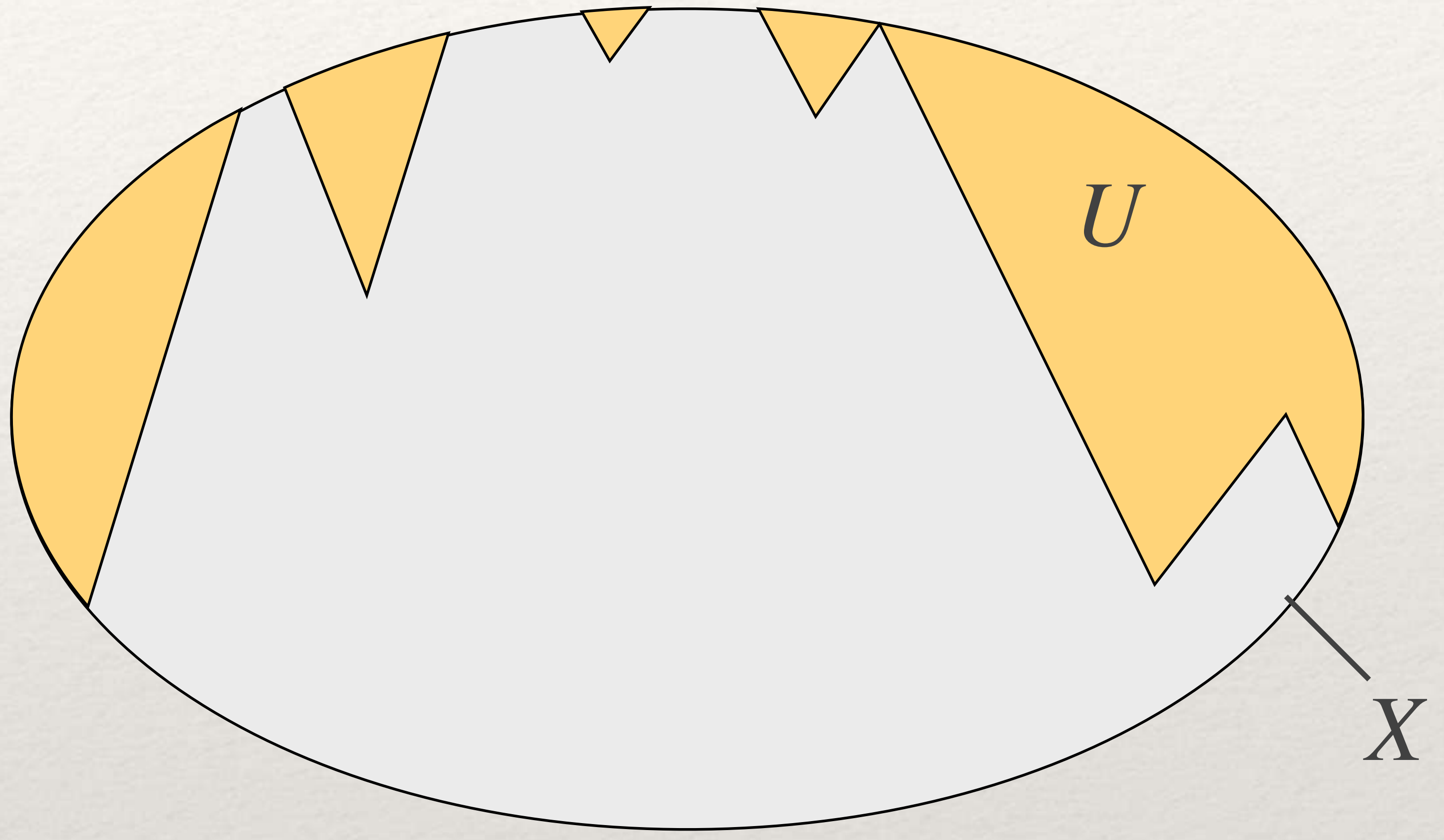
- A closed set $F \in \mathscr{H}X$ is **irreducible** iff
  for all $F_1, \cdots, F_n \in \mathscr{H}X, F \subseteq \bigcup_i F_i \Rightarrow \exists i, F \subseteq F_i$
- Every set $\downarrow x$ is irreducible closed
  X is **sober** iff T_0
  and those are the only irreducible closed sets
- The **sobrification** $X^s = \{F \in \mathscr{H}X \mid F \text{ irreducible}\}$,
  seen as subspace of $\mathscr{H}X$ is always **sober**,
  and X embeds into X^s through $x \mapsto \downarrow x$
- X and X^s have isomorphic lattices of open subsets

E.g., $\mathscr{H}X$, Spec(R),
but not $\mathbb{N}_{cof}$, X^* for example

In particular,
X Noetherian iff X^s Noetherian

Representing open sets: the trick

- Embed state space X into its **sobrification** X^s



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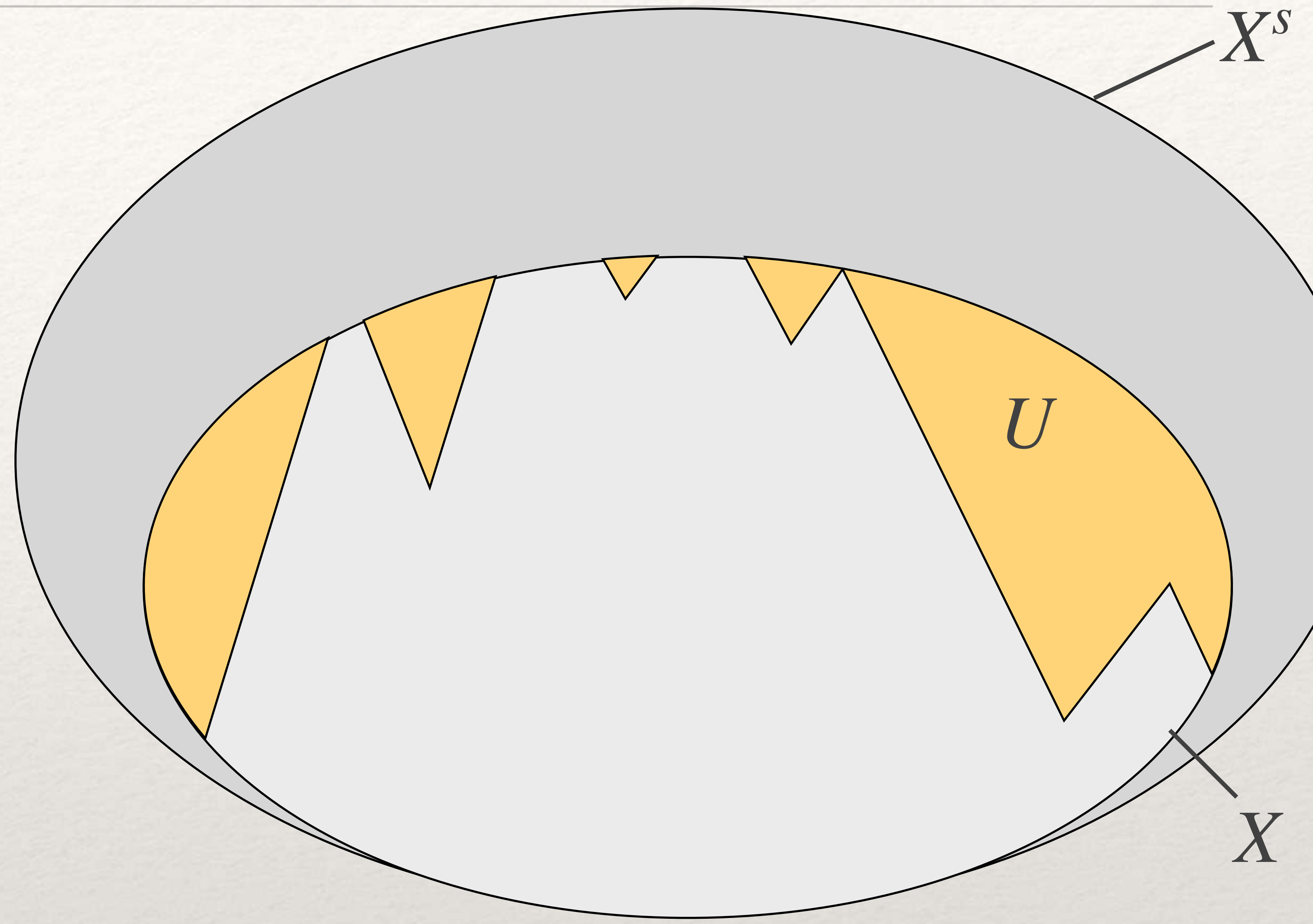

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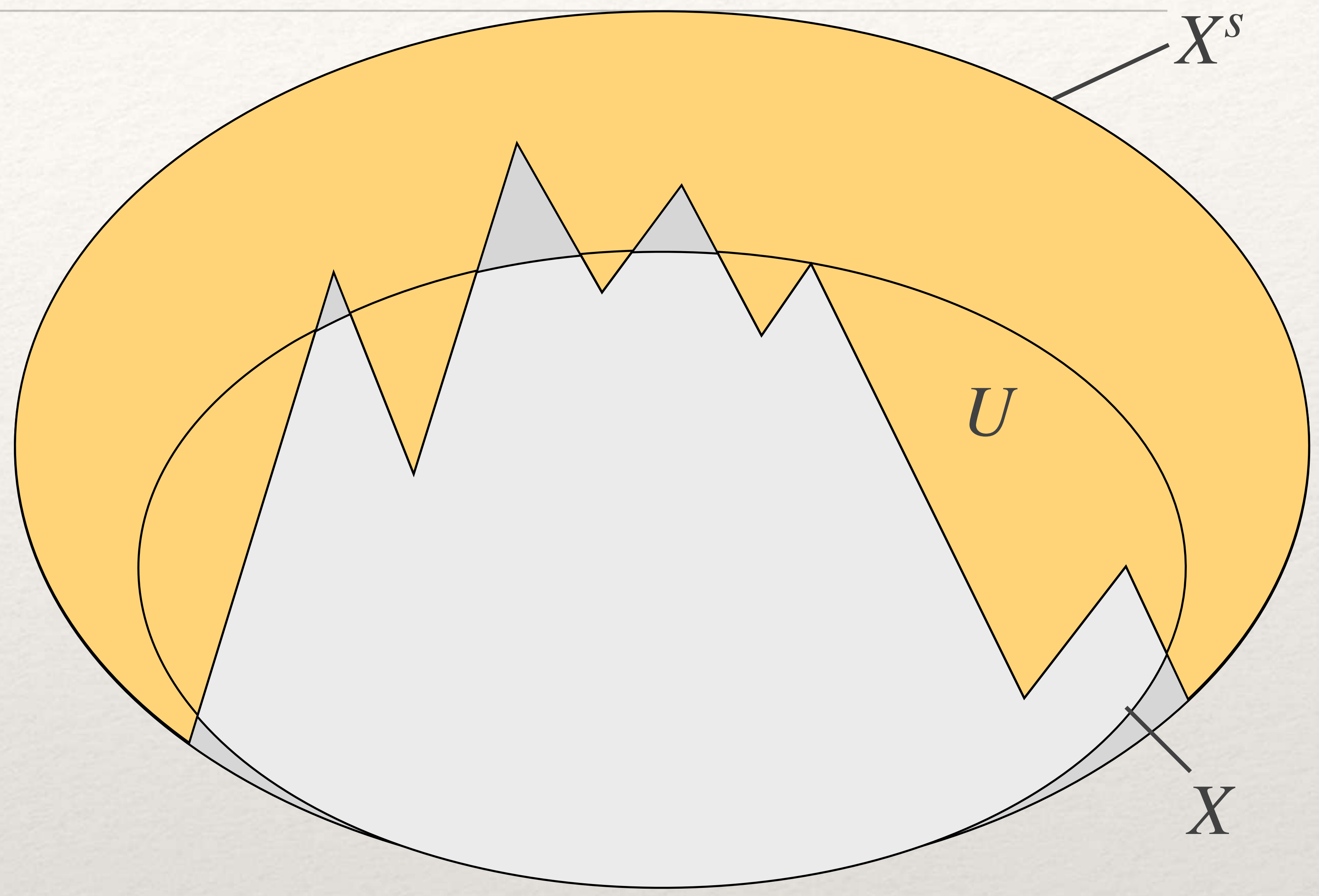

Representing open sets: the trick

- Embed state space X into its **sobrification** X^s
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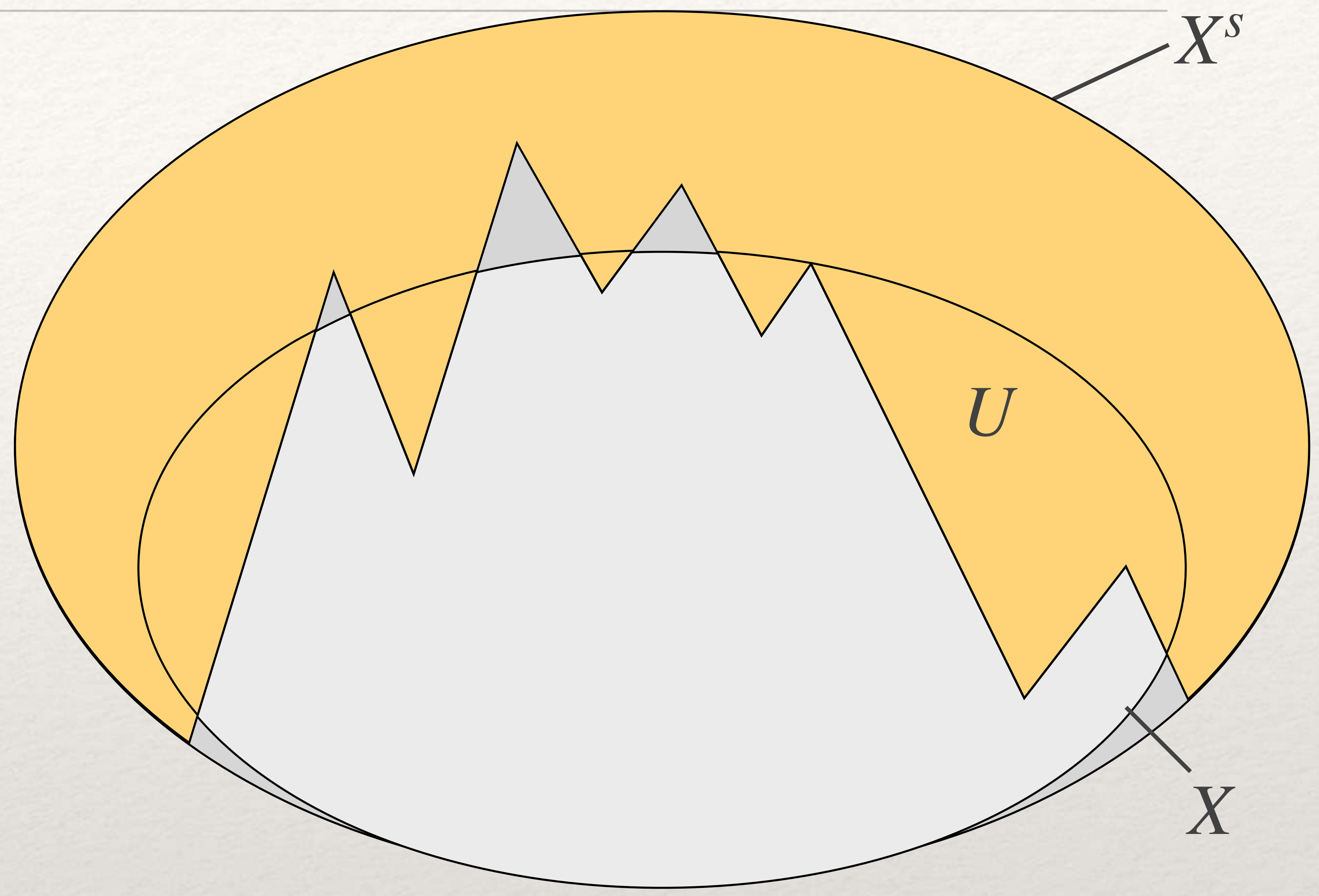
- Represent open sets U by their complements: **closed sets** C



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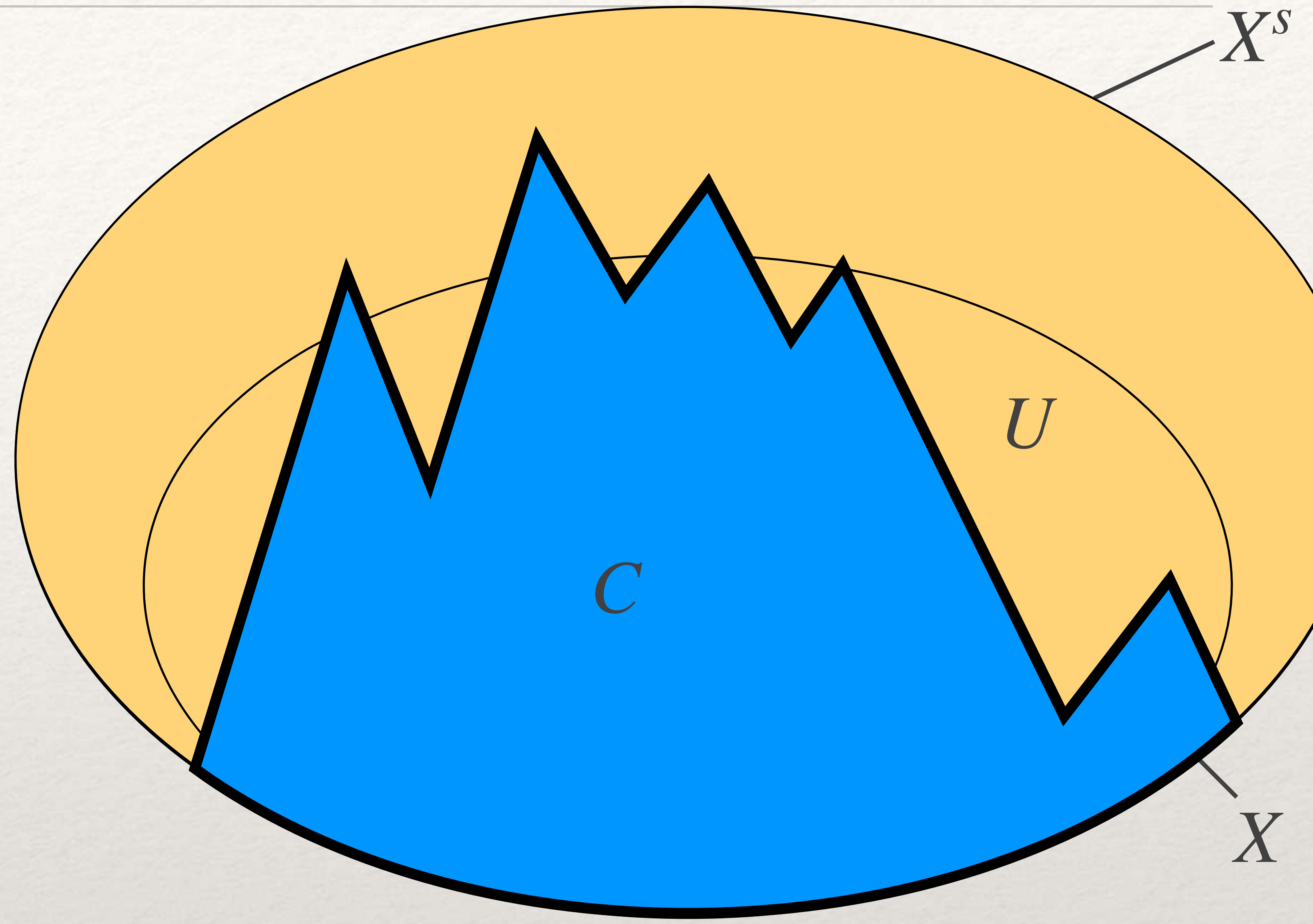

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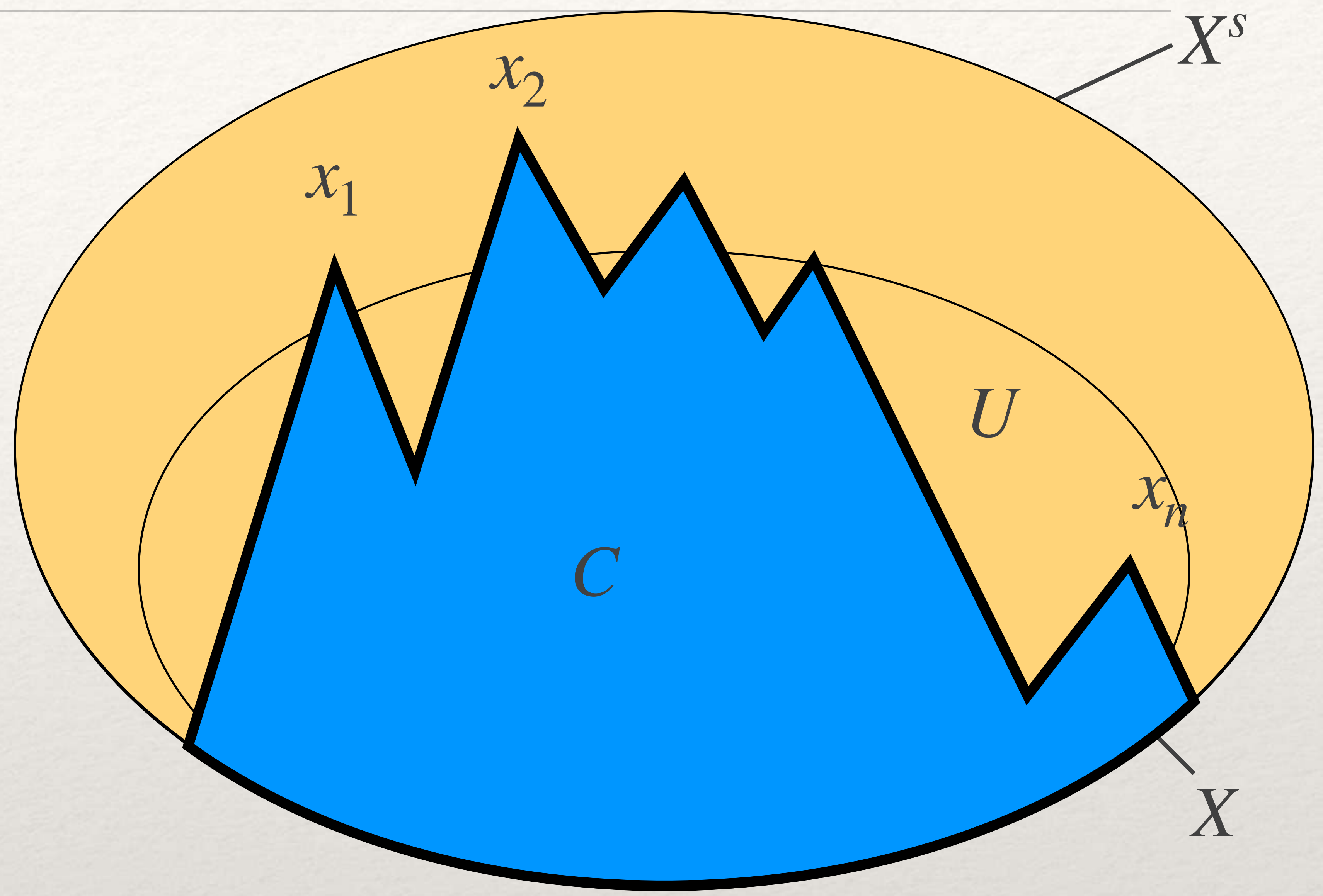

- Now:

In a sober Noetherian space, every closed set C is a **finitary** subset $\downarrow\{x_1, \cdots, x_n\}$.

Reminder

- **Proposition.** If:
  — X is **well-founded**
  — (Property T) X is **finitary**
  — (Property W) For all $x, y \in X$, $\downarrow x \cap \downarrow y$ is **finitary**
  then X is **Noetherian** in the upper topology and the **closed** sets are exactly the **finitary subse**

This turns out to be the general form of all **sober** Noetherian spaces.

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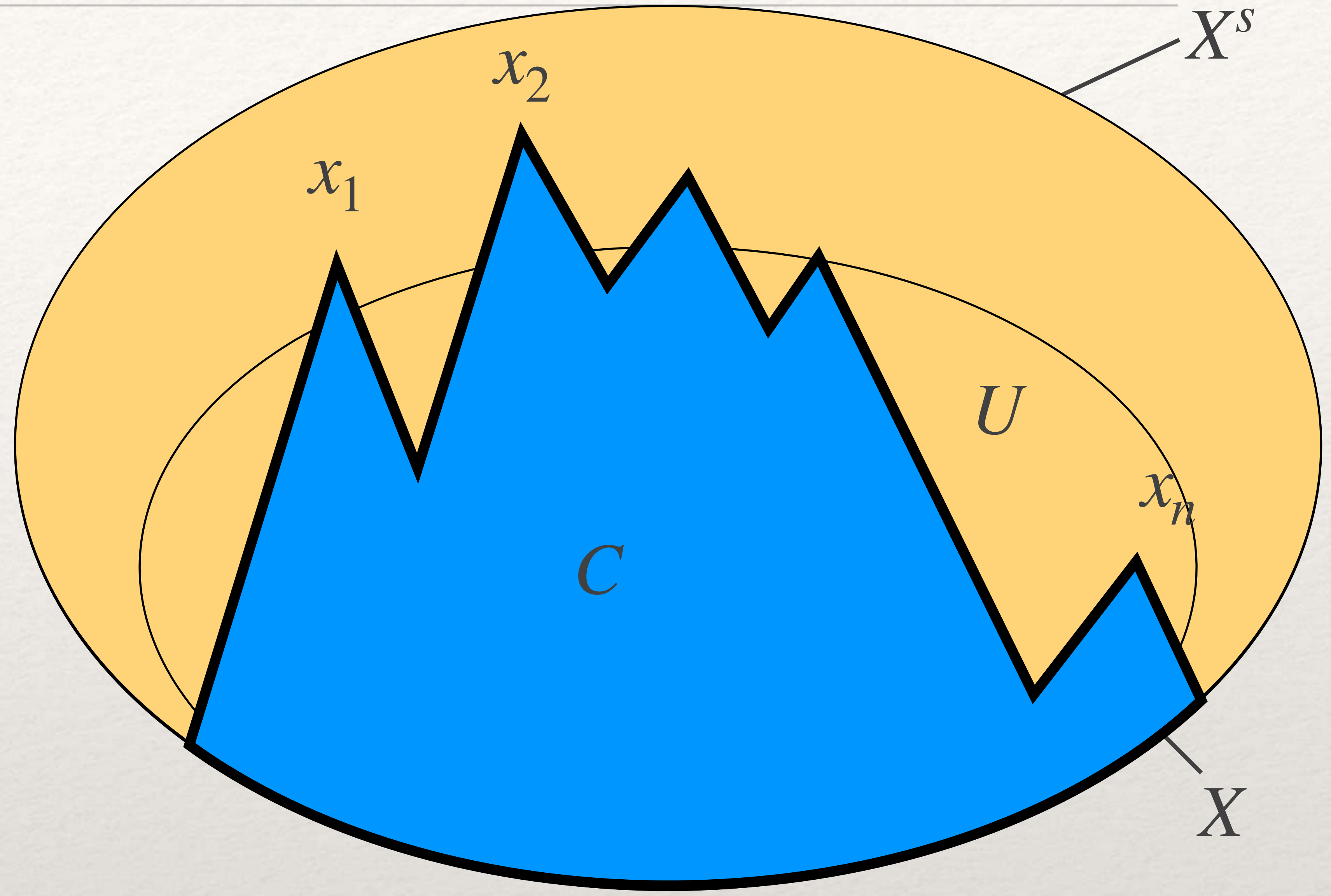

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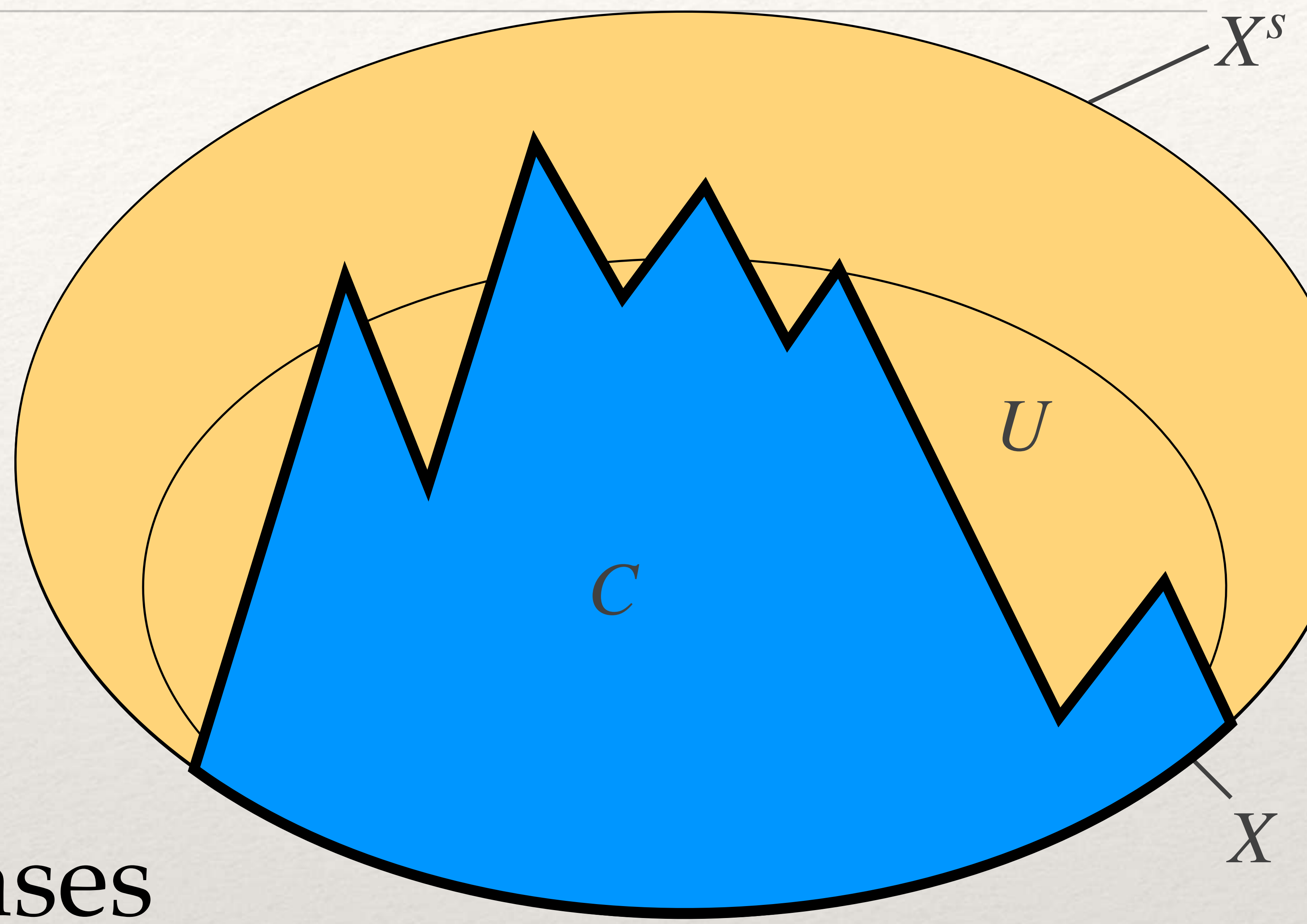
This turns out to be the general form of all **sober** Noetherian spaces.

- Hence we can represent U by
(the complement of the downward closure in X^s)
of **finitely many** points… in X^s

Representing points in sobrifications

- For a finite set Σ, with the discrete topology,
  $\Sigma^s = \Sigma$
- Products: $(X \times Y)^s = X^s \times Y^s$
- Spec($\mathbb{Z}[X_1, \cdots, X_n]$): already sober,
  points = prime ideals, represented as Gröbner bases
- $(X^*)^s$ consists of **word products**
  $$P ::= \epsilon \mid C^? P \mid F^* P$$
  with $C \in X^s$, $F = C_1 \cup \cdots \cup C_n$ ($C_i \in X^s$)



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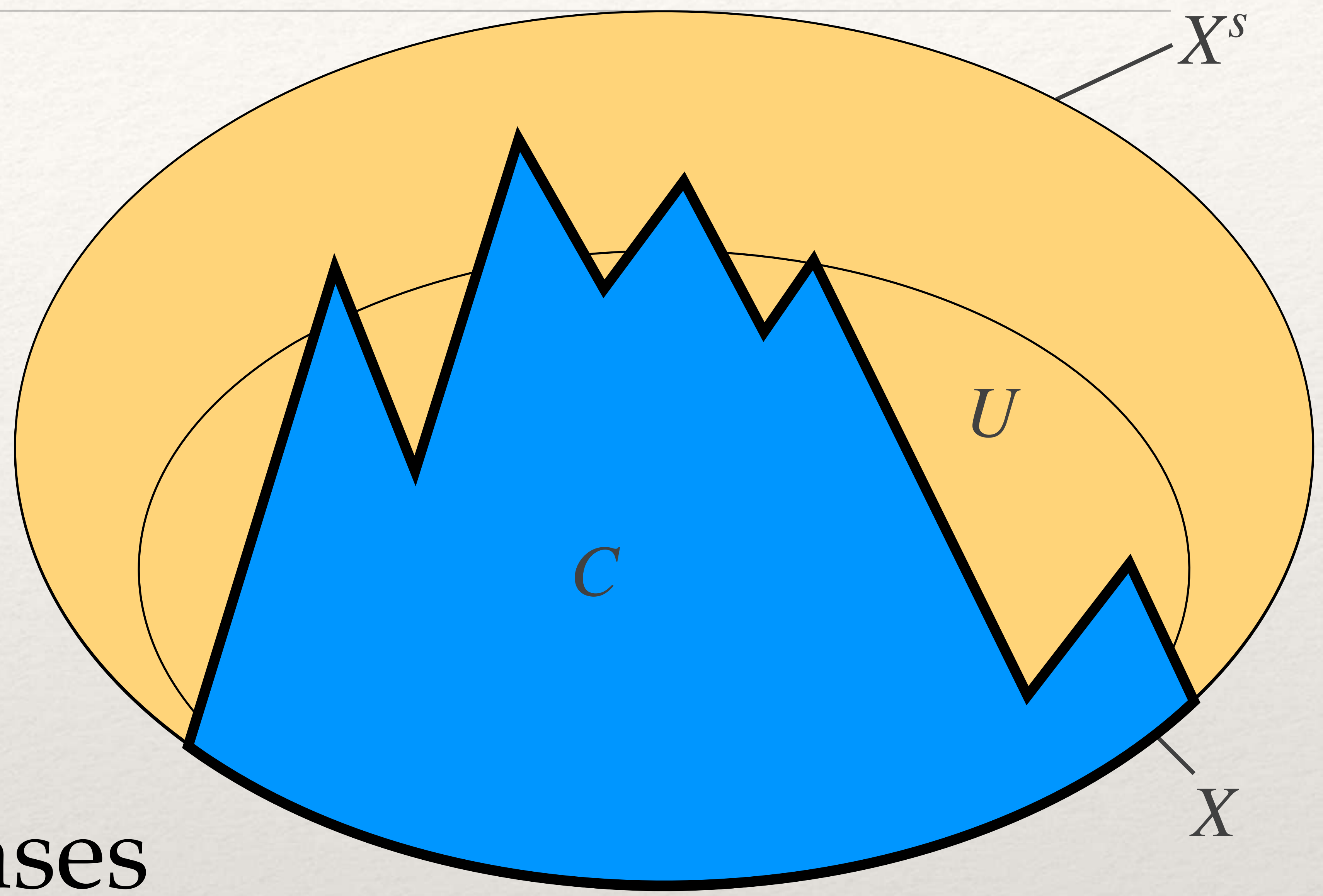

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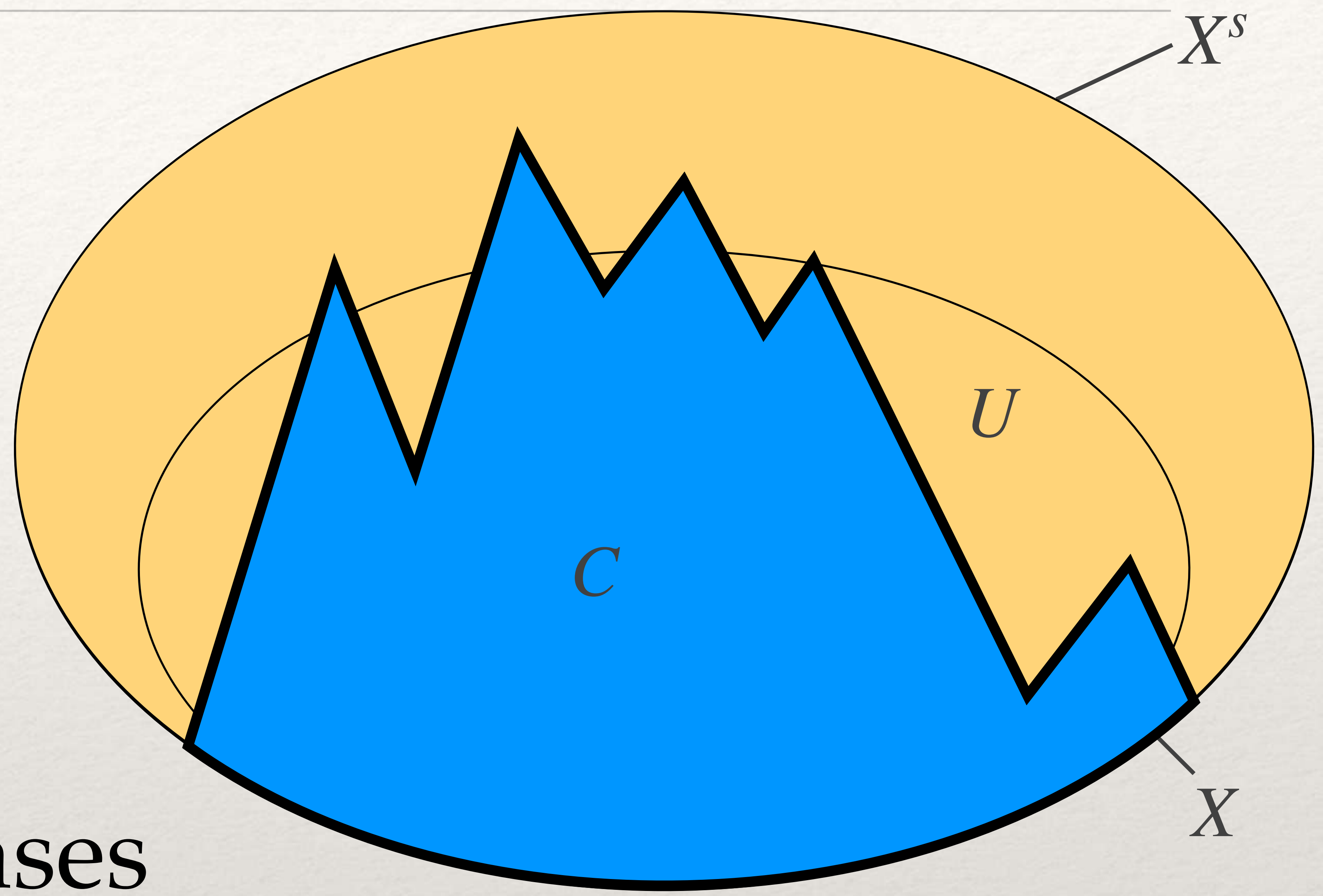

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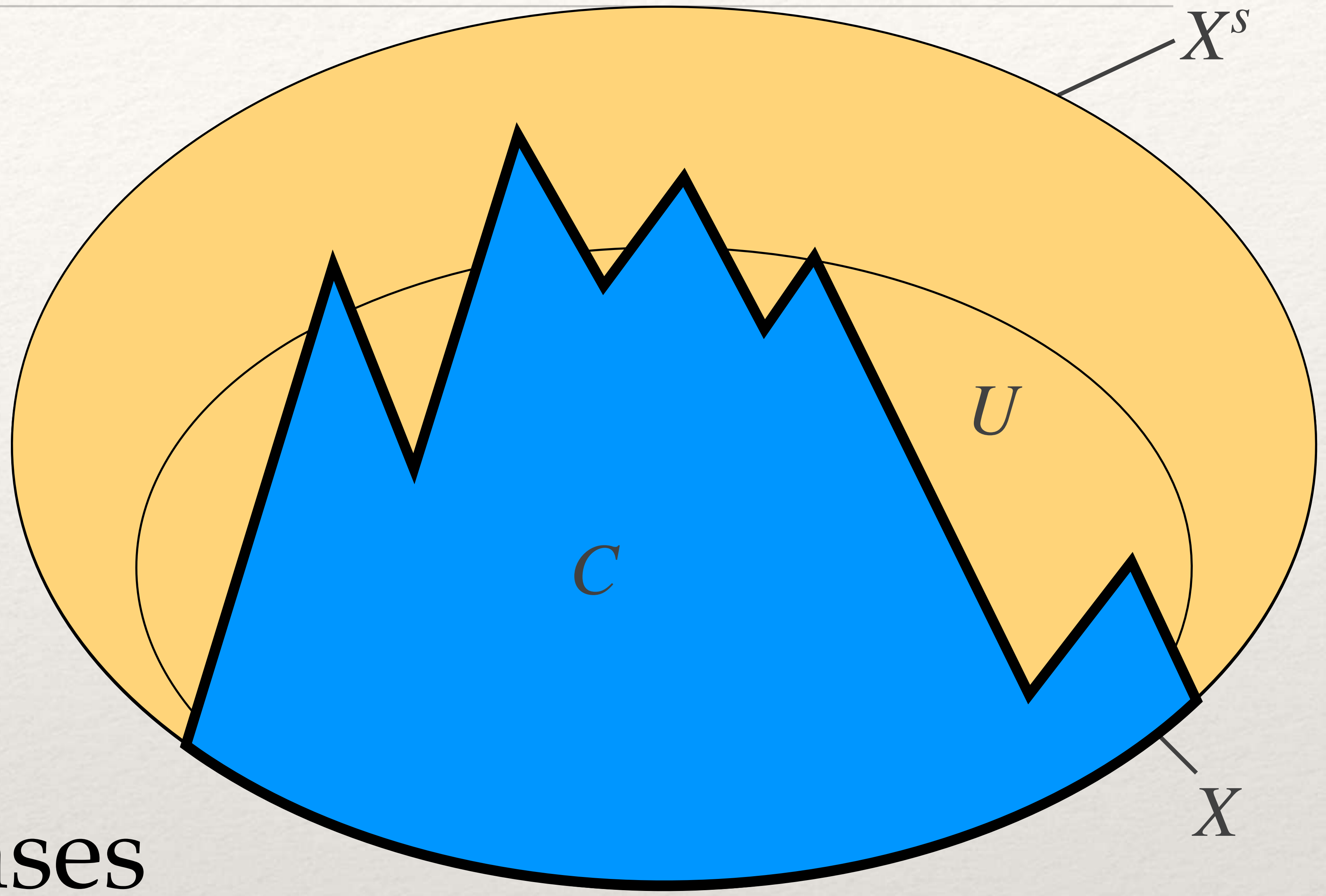

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Other word products, e.g., $C_1^? C_2^? F_1^* C_3^? F_2^* F_3^*$

- All those are **representable** on a computer (Finkel, JGL 2009, 2021)

Statures of Noetherian spaces

- **Maximal order types** of well-partial-orderings
- **Statures** of Noetherian spaces as generalization of maximal order types
- … we are not really changing the subject,
  and we will use the **representations** of points in X^s again

Maximal order types

- A well-partial-ordering is a well-quasi-ordering that is antisymmetric
- **Theorem** (Wolk 1967). A wpo is a partial ordering whose linear extensions are all **well-founded**

  Note: every linear well-founded ordering is isomorphic to a unique ordinal, … its **order type**

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- **Theorem** (de Jongh, Parikh 1977). Among those, one has **maximal** order type.
- Any meaningful equivalent of that notion for Noetherian spaces?
  But first, why should we bother about maximal order types anyway?

Why bother about maximal order types?

- First studied by de Jongh and Parikh (1977)
  then Schmidt (1979)

- Many applications in proof theory (reverse mathematics):
  Simpson (1985), after Friedman
  van den Meeren, Rathjen, Weiermann (2014, 2015)
  etc.

- Ordinal complexity of the size-change principle for proving the termination of programs and rewrite systems
  Blass and Gurevich (2008)

- and…

Why bother about maximal order types?

- Figueira, Figueira, Schmitz and Schnoebelen (2011),
  Schmitz and Schnoebelen (2011) (and others)

  obtain **complexity upper bounds**
  for algorithms whose termination
  is based upon wqo arguments
  (e.g., coverability)

Theorem 5.3 (Main Theorem).
Let g be a smooth control function
eventually bounded by a function in $\mathscr{F}_{\gamma}$,
and let A be an exponential nwqo
with maximal order type $< \omega^{\beta+1}$.
Then $L_{A,g}$ is bounded by a function in:

- $\mathscr{F}_{\beta}$ if $\gamma < \omega$ (e.g., if g is primitive recursive) and $\beta \geq \omega$
- $\mathscr{F}_{\gamma+\beta}$ if $\gamma \geq 2$ and $\beta < \omega$.

length function
(complexity upper bound)

class of functions
elementary recursive in F_{β}
(fast-growing hierarchy)

From S. Schmitz, Ph. Schnoebelen, *Multiply-recursive upper bounds with Higman's Lemma*. ICALP 2011.

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- E.g., coverability
  in lossy channel systems
  is F_{ω^ω}**-complete.**
  (way larger than Ackermann)

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Going topological

- Let us return to the question of finding a **Noetherian analogue** of maximal order types

A wrong idea: minimal T_0 topologies

- Partial ordering ~ T_0 topology
  Extension ~ coarser T_0 topology
  Linear extension = maximal extension ~ minimal T_0 topology
- Studied by Larson (1969).
  A minimal T_0 topology is necessarily the **upper** topology of a **linear** ordering.
- Unfortunately, minimal T_0 topologies do not exist in general:
  Fact. $\mathbb{R}_{cof}$ is Noetherian, but has no coarser minimal T_0 topology.
  (Its uncountably many proper closed subsets would all have to be finite, and linearly ordered.)

Statures of wpos

- **Theorem** (Kříž 1997, Blass and Gurevich 2008).
  Maximal order type of a wpo $(X, \leq)$
  $=$ **ordinal rank** $||X||$ of the top element X
  in the poset $(\mathscr{D}X, \subseteq)$ of downwards-closed subsets of X

The **stature** of X

- Ordinal rank inductively defined by:
  $$||F|| = \sup\{\, ||F'|| + 1 \mid F' \in \mathscr{D}X, F' \subsetneq F \}$$

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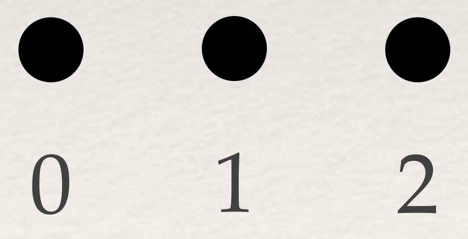

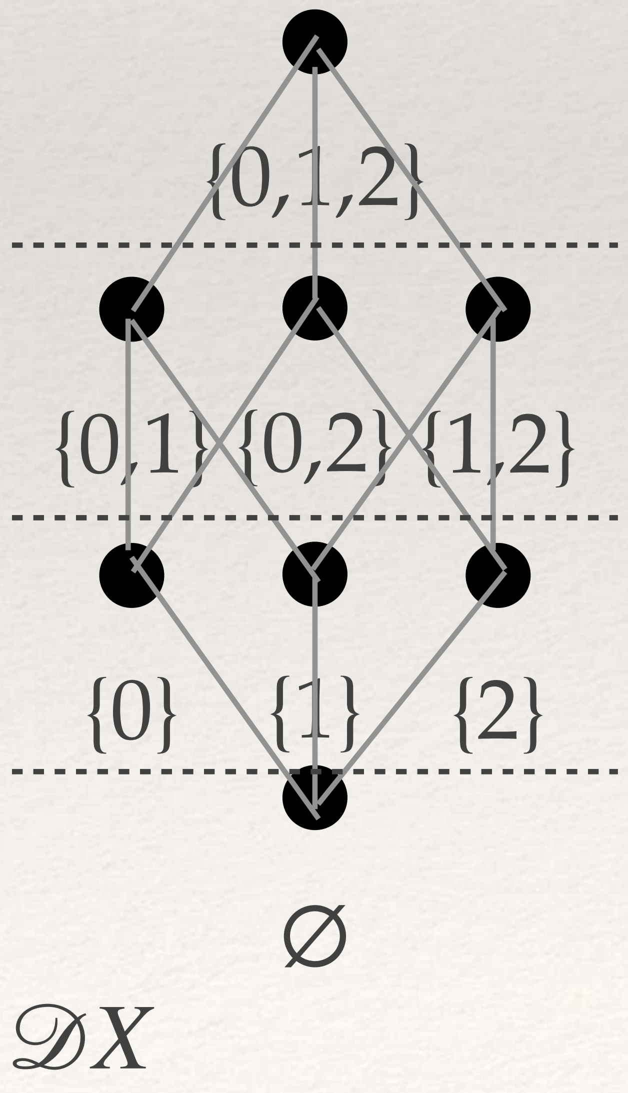

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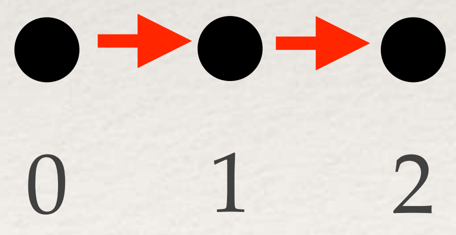
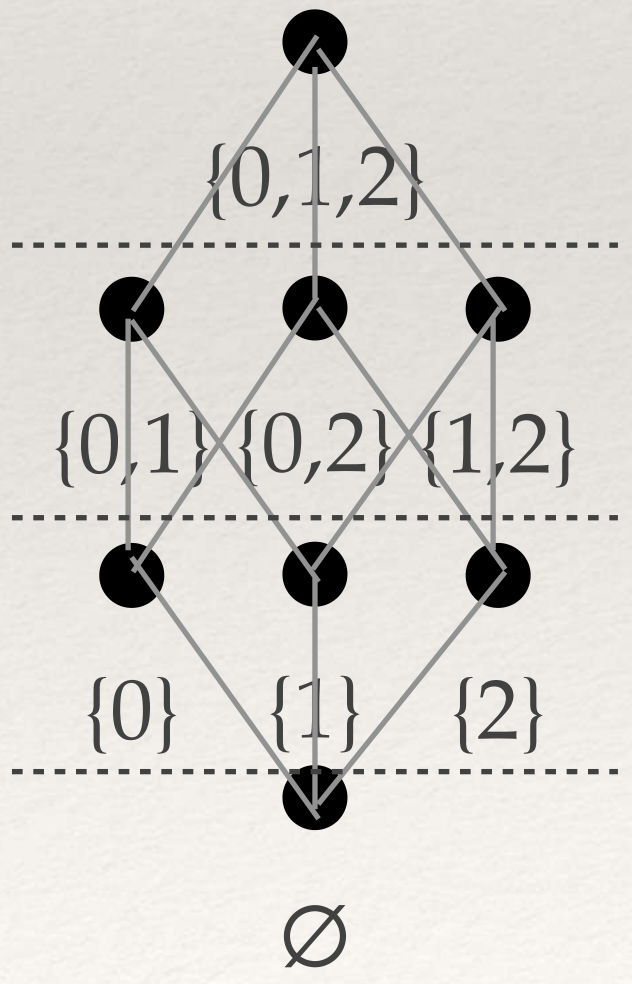


maximal order type=3



$\mathscr{D}X$

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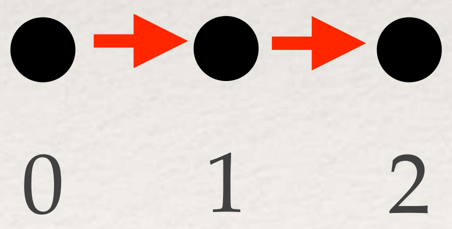

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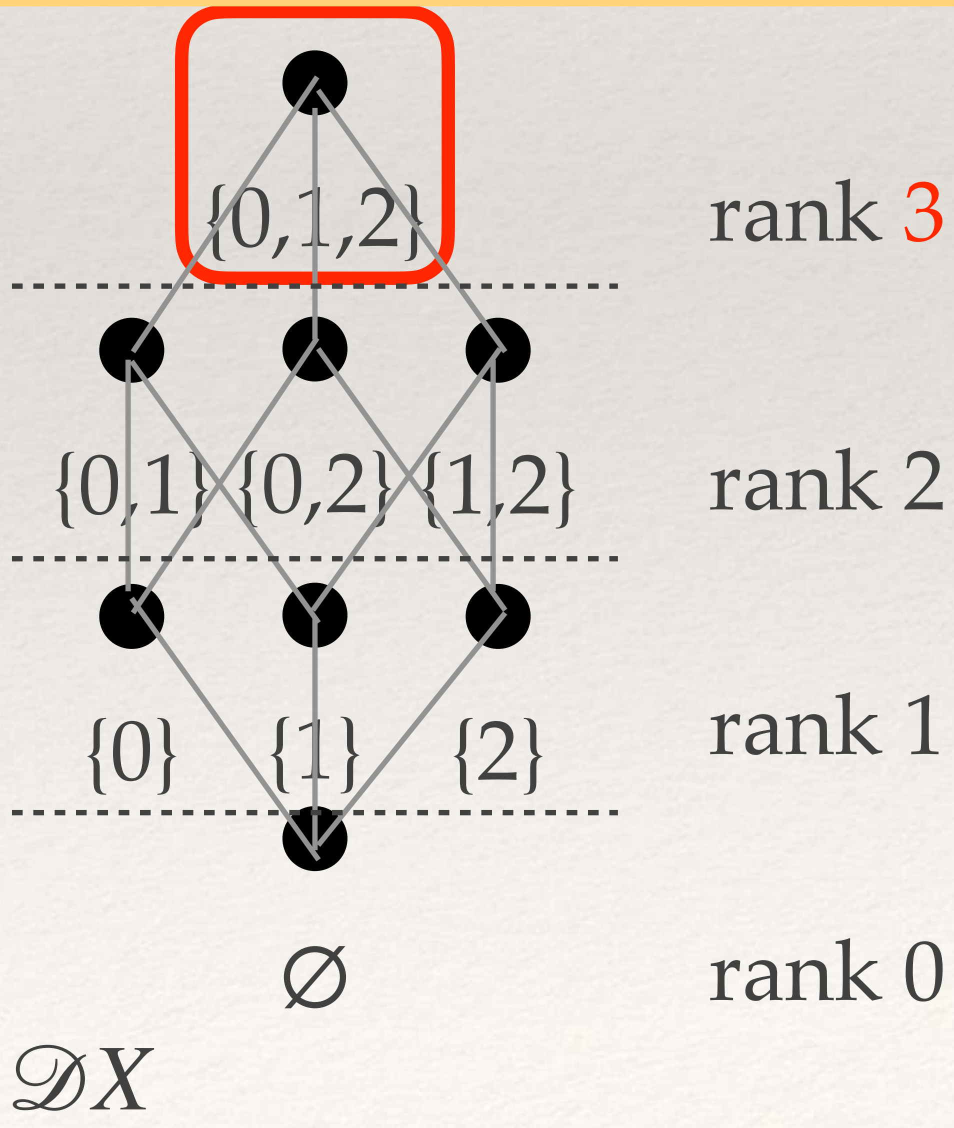

Statures of Noetherian spaces

- **Definition.** The **stature** of a Noetherian space X is the **ordinal rank** $||X||$ of the top element X in the poset $(\mathcal{H}X, \subseteq)$ of **closed** subsets of X

- $$||F|| = \sup\{\, ||F'|| + 1 \mid F' \in \mathcal{H}X, F' \subsetneq F\}$$

- Matches previous definition:
  for a wqo in its Alexandroff topology,
  closed = downwards-closed
  $\mathcal{H}X = \mathcal{D}X$

X is Noetherian iff:
(6) Every antitonic chain $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ of closed subsets is **stationary**
(7) $\mathcal{H}X$ is well-founded.

Some statures of Noetherian spaces

- We have already obtained **statures** of quite a few Noetherian constructions (JGL, Laboureix 2022)
- Let me focus on X^*

X	sob X		$\Vert X\Vert$	
Finite T_0	$\leq$ card X		card X	Lem. 6.1
Ordinal α (Alex.)	α / $\alpha+1$	Lem. 6.2	α	Lem. 6.2
Ordinal α (Scott)	α / $\alpha+1$	Lem. 6.2	α / $\alpha-1$	Lem. 6.2
Cofinite topology	1 / 2	Thm. 7.1	min(card X, ω)	Thm. 7.2
$X+Y$	max(sob X, sob Y)	Prop. 8.4	$\Vert X\Vert \oplus \Vert Y\Vert$	Prop. 8.2
$X +_{\text{lex}} Y$	sob X + sob Y	Prop. 9.4	$\Vert X\Vert + \Vert Y\Vert$	Prop. 9.2
$X_\perp$	$1+$ sob X	Prop. 9.6	$1+\Vert X\Vert$	Prop. 9.6
$X \times Y$	(sob $X \oplus$ sob Y) -1	Prop. 10.1	$\Vert X\Vert \otimes \Vert Y\Vert$	Thm. 10.9
$\mathcal{H}_{0\mathsf{V}} X$, $\mathcal{H}_{\text{fin}} X$, $\mathbb{P}X$, $\mathbb{P}_{\text{fin}} X$	$\Vert X\Vert + 1$	Thm. 11.1	$\geq 1+\Vert X\Vert$, $\leq \omega^{\Vert X\Vert}$	Prop. 11.2
X^*	$\omega^{\Vert X\Vert^\circ}+1$ ($\alpha^\circ \stackrel{\text{def}}{=} \alpha+1$ if $\alpha=\epsilon+n$, ϵ critical, $n \in \mathbb{N}$, α otherwise)	Thm. 12.13	$\omega^{\omega^{\Vert X\Vert'}}$ ($\alpha' \stackrel{\text{def}}{=} \alpha-1$ if α finite, α° otherwise)	Thm. 12.22
$\triangleright_{n=1}^{+\infty} X_n$	$\bigoplus_{n=1}^{+\infty}$ rsob $X_n + 1$ / $\bigoplus_{n=1}^{k}$ rsob $X_n + \omega + 1$	Thm. 13.4	$\bigotimes_{n=1}^{+\infty} \Vert X_n\Vert$ / $\bigotimes_{m=1}^{k} \Vert X_m\Vert \times \omega$	Thm. 13.8
$X^\triangleright$	$\omega^{\alpha_1+1}+1$ where sob $X - 1 =_{\text{CNF}} \omega^{\alpha_1} + \cdots$	Cor. 13.7	$\omega^{\omega^{\beta_1+1}}$ / ω where $\Vert X\Vert =_{\text{CNF}} \omega^{\alpha_1}+\cdots$, $\alpha_1 =_{\text{CNF}} \omega^{\beta_1}+\cdots$	Cor. 13.9
$X^{\circledast}$	$\geq (\omega \times \Vert X\Vert)+1$, $\leq (\Vert X\Vert \otimes \omega)+1$	Prop. 14.8, Prop. 14.9	$\omega^{\widehat{\alpha}}$ ($\widehat{\alpha} \stackrel{\text{def}}{=} \omega^{\alpha_1{}^\circ} + \cdots + \omega^{\alpha_m{}^\circ}$ if $\alpha =_{\text{CNF}} \omega^{\alpha_1}+\cdots+\omega^{\alpha_m}$)	Thm. 14.20

The stature of X*

- **Theorem** (JGL, Laboureix 2022). If $X \neq \emptyset$ is Noetherian and $\alpha = ||X||$, then $||X^*|| = \omega^{\omega^{\alpha \pm 1}}$
  (+1 if $\alpha = \epsilon_\beta + n$, –1 if α finite)
- Not very surprising: already known when X wqo (Schmidt 1979)

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- The proof is very different, and is **localic**.
  Explicitly, we do not reason on points (words),
  but on **closed sets** = finite unions of word products

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An excerpt from the proof of $||X^*|| \geq \omega^{\omega^{\alpha \pm 1}}$

- Let $F \subsetneq F \cup C$, $\mathbf{C}_0 = \emptyset$, $\mathbf{C}_{n+1} = (F^*C^?)^n F^*$, $\mathscr{A}_n = \{\mathbf{A} \in \mathscr{H}X \mid \mathbf{C}_n \subseteq \mathbf{A} \subsetneq \mathbf{C}_{n+1}\}$
- Map $(\mathbf{B} \subsetneq \mathbf{B}^+) \in \mathrm{Step}(\mathscr{H}(F^*))$, $\mathbf{A} \in \mathscr{A}_n$ to $(F^*C^?)^{n+1}\mathbf{B} \cup \mathbf{A}C^?\mathbf{B}^+ \cup \mathbf{C}_{n+1}$

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A finite union
of word products

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- If $||F^*|| \geq \omega^{\omega^\beta}$ then $||\mathbf{C}_{n+1}|| \geq \omega^{\omega^\beta \times (n+1)}$,
  so $||(F \cup C)^*|| \geq \omega^{\omega^{\beta+1}}$, by taking suprema over $n \in \mathbb{N}$
- This is the key step in a well-founded induction on $F \in \mathscr{H}X$
  showing $||F^*|| \geq \omega^{\omega^{||F|| \pm 1}}$
- Finally, let $F = X$; by definition, $||X|| = \alpha$. $\square$

A finite union of word products

The stature of $\mathbb{Z}[X_1, \cdots, X_n]$

- The ordinal height of the lattice of ideals of $\mathbb{Z}[X_1, \cdots, X_n]$ is $\omega^n + 1$ (Aschenbrenner, Pong 2004)
- Hence $||\mathrm{Spec}(\mathbb{Z}[X_1, \cdots, X_n])|| = \omega^n$ (argument not quite written out yet, probably well-known)

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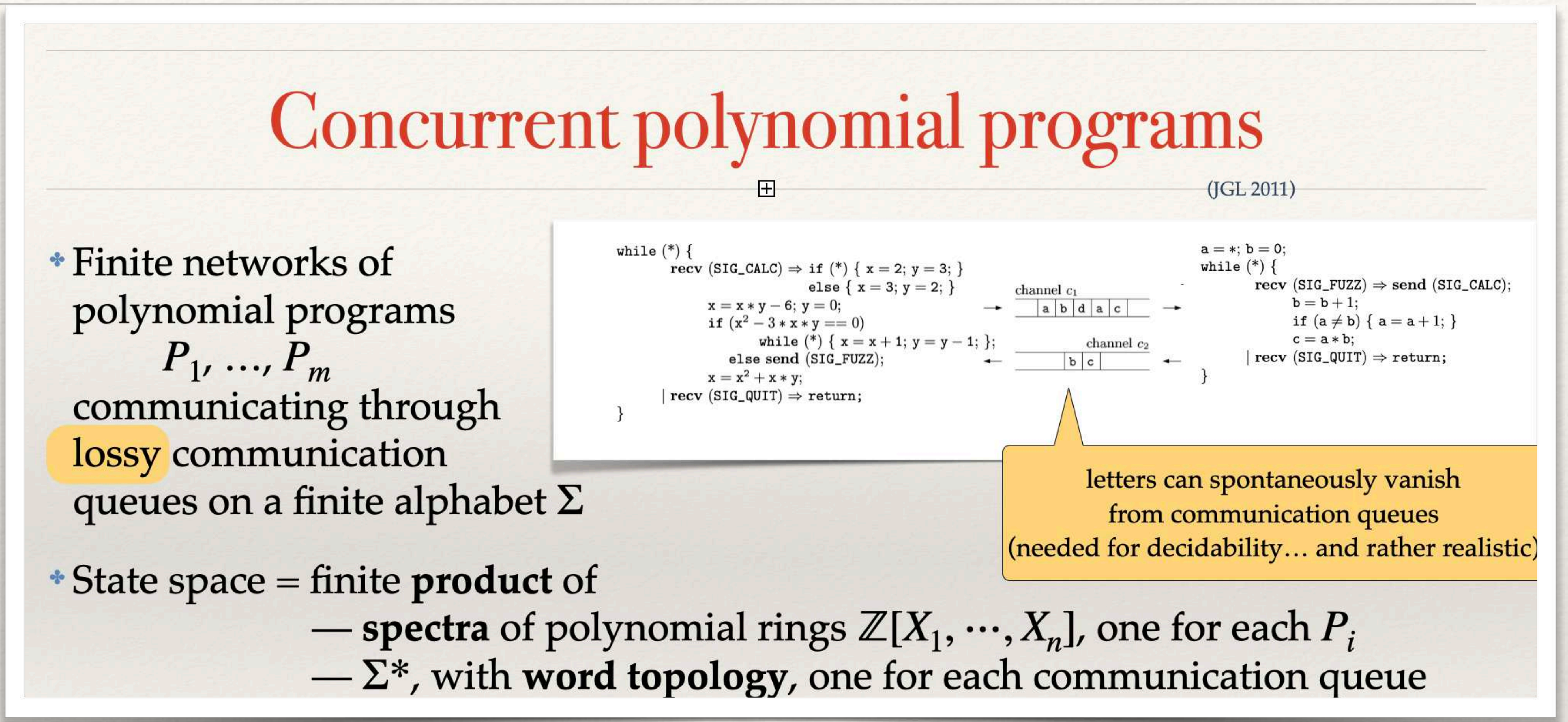
- Together with $||X \times Y|| = ||X|| \otimes ||Y||$ (JGL, Laboureix 2022) extending the same formula on wqos (de Jongh, Parikh 1977), we obtain the **stature** of the state space of **concurrent polynomial programs**…

The stature of the state space of concurrent polynomial programs

- m programs, each on n variables
  p queues, on $k \geq 1$ letters

- Stature of state space =
  $$(\omega^n)^m \otimes (\omega^{\omega^{k-1}})^p$$
  $$= \omega^{nm \oplus \omega^{k-1}.p}$$



- Note that the contribution of the polynomial programs (nm)
  is **much lower** than the contribution of the queues ($\omega^{k-1}.p$)

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ordinal α (Alex.)	α
$Y+Z$	$\|\|Y\|\| \oplus \|\|Z\|\|$
$Y+_{lex}Z$	$\|\|Y\|\| + \|\|Z\|\|$
$Y_\perp$	$1+\|\|Y\|\|$
$Y\times Z$	$\|\|Y\|\| \otimes \|\|Z\|\|$
fin. words Y^*	ω^{$\omega^{\|\|Y\|\|\pm 1}$}
multisets $Y^\circledast$	$\omega^{\tilde{\alpha}}$ [$\|\|Y\|\|=\alpha$]

From JGL and B. Laboureix, *Statures and sobrification ranks of Noetherian spaces*. Submitted, 2022. https://arxiv.org/abs/2112.06828

Bottom row from JGL, S. Halfon, and A. Lopez, *Infinitary Noetherian Constructions II. Transfinite Words and the Regular Subword Topology*. Submitted, 2022. https://arxiv.org/abs/2202.05047

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ordinal α (Scott)	α / $\alpha-1$
cofinite topology	min (card Y, ω)
$\mathcal{H}Y$, $\mathbb{P}Y$	$1+\|\|Y\|\| \ldots \omega^{\|\|Y\|\|}$
words, prefix top.	ω^{$\omega^{\beta+1}$} [$\|\|Y\|\|=\omega$^{$\omega^{\beta}+\ldots$} + …]
$Y^{<\alpha}$	$\leq \omega$^{$\omega^{(\|\|Y\|\|+\alpha)\pm 1}$}

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fin. words Y^*	ω^{$\omega^{\Vert Y \Vert \pm 1}$}	$\omega^{\Vert Y \Vert \pm 1}$
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- Missing: finite **trees**, notably (see Schmidt 1979 for the wqo case)
- Application to actual **complexity** upper bounds?

statures | sobrification ranks

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Conclusion

Conclusion, research directions

- A rich theory extending **wqos** into the topological: **Noetherian** spaces
- Old results extend, new results pop up (powersets, spectra, infinite words)
- Ordinal analysis: the **stature** $||X||$=ordinal rank of top element of $\mathscr{H}X$ as an analogue of maximal order types
- Still in its infancy