Generalized subspaces in the duality of T_D-spaces

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Parts of the talk are joint work with Javier Gutiérrez García and Anna Laura Suarez

Background

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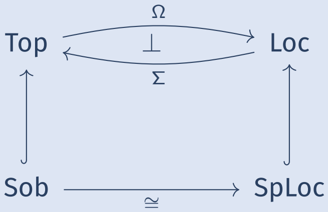

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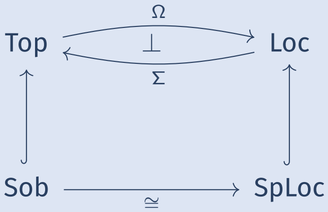

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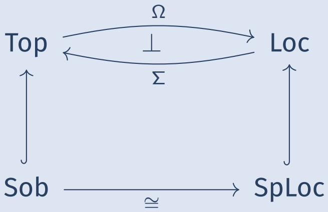

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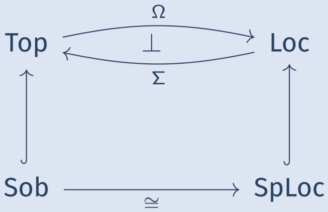

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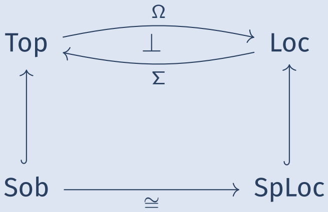

Sobriety is an important property in the theory of locales (e.g., it allows one to reconstruct a space from its lattice of open sets). But there is also the equally important **T_D-axiom**. A space X is T_D if for every $x \in X$, there is an open $x \in U$ with $U - \{x\}$ open.

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In this context, it is natural whether there is a similar categorical framework as the classical duality between spaces and frames.

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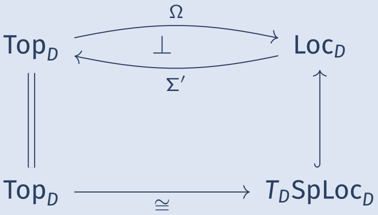

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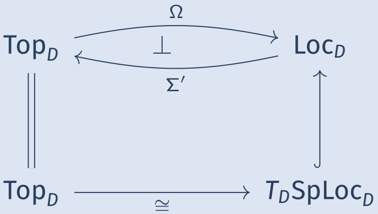

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Because Ω is full and faithful, we may regard Loc_D has a category of generalized T_D-spaces.

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- Their left adjoints are called **frame D-homomorphisms**.
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- We will also consider its dual category, Frm_D, of frames and D-homomorphisms.

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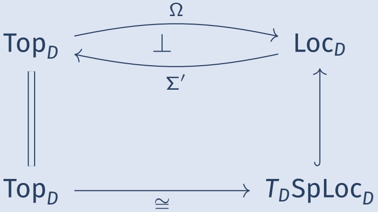

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- I. A., J. Gutiérrez García, On the categorical behaviour of locales and D-localic maps, *Quaestiones Mathematicae* (2022).
- I. A., A.L. Suarez, The coframe of D-sublocales and the T_D-duality, *Topology and its Applications* (2021).

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A sublocale $S \subseteq L$ is a ***D*-sublocale** if $\mathrm{pt}_D(S) \subseteq \mathrm{pt}_D(L)$ — i.e., if and only if the embedding $S \hookrightarrow L$ belongs to Loc_D.

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The regular subobjects of a locale L in Loc *are precisely the sublocales of L.*

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The system $\mathcal{S}(L)$ of all sublocales of L is a coframe, and the map

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is a frame homomorphism.

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is a frame D-homomorphism.

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$\mathcal{S}_D(L)$ is a dense[1] subcolocale of $\mathcal{S}(L)$. In particular, it is a zero-dimensional coframe.

[1] Shorthand for "$\mathcal{S}_D(L)^{op}$ is a dense sublocale of $\mathcal{S}(L)^{op}$"

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2. Every pointless sublocale is a D-sublocale.
3. The diagonal sublocale is a D-sublocale of the square $L \oplus L$.

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Say that a frame homomorphism **lifts** if there is another frame homomorphism $\mathcal{S}_D(f)$ such that the following square commutes:

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The assignment $L \mapsto \mathcal{S}(L)^{op}$ can be made into a functor $\mathcal{S}(-)^{op} \colon \mathrm{Frm} \to \mathrm{Frm}$ naturally — i.e., for every frame homomorphism f, the following square is commutative:

$$\begin{array}{ccc} \mathcal{S}(L)^{op} & \xrightarrow[\mathcal{S}(f)^{op}]{} & \mathcal{S}(M)^{op} \\ \uparrow{\scriptstyle \mathfrak{c}_L} & & {\scriptstyle \mathfrak{c}_M}\uparrow \\ L & \xrightarrow{f} & M \end{array}$$

T_D-duality

Say that a frame homomorphism **lifts** if there is another frame homomorphism $\mathcal{S}_D(f)$ such that the following square commutes:

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- **Surjections**: A frame surjection $L \twoheadrightarrow S$ lifts iff it is a D-homomorphism.
- **Monomorphisms**: However, one can construct an open D-homomorphism between T_D-spatial frames which does not lift.

The construction $\mathcal{S}_D(L)$ is not functorial in general.

An application to T_D-spatiality

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A frame L is said to be **totally spatial** if every sublocale of L is spatial.

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Theorem (Niefield-Rosenthal)

The following are equivalent for a frame L:

(i) *L is totally spatial.*

(ii) *$\mathcal{S}(L)^{op}$ is spatial.*

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Example

The Alexandroff topology on the natural numbers is totally T_D-spatial.

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Example

The Alexandroff topology on the natural numbers is totally T_D-spatial. But not every prime is covered.

This situation cannot happen in the T_1-case. If every sublocale is T_1-spatial, then every prime is automatically maximal!

Thank you!