Hereditary Structural Completeness over K4: Rybakov's Theorem Revisited

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Background

Definition A rule Γ/φ is said to be admissible for a deductive system $\vdash$ iff the set of tautologies of $\vdash$ is closed under applications of Γ/φ.
It is derivable for $\vdash$ iff $\Gamma \vdash \varphi$.

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Whilst every derivable rule for a given deductive system is admissible the converse can fail.

This gap has motivated an in depth study of admissibility, including Friedman [1975], Rybakov [1984], Iemhoff [2001] and Jeřábek [2010].

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Investigations by Prucnal [1972] and Dzik & Wroǹski [1973] among others suggested that whilst a full characterisation of SC intermediate and modal logics was out of reach a herediarily strucutrally complete (HSC) characterisation might be possible.

Definition If every finitary extension of $\vdash$ is structurally complete then we say $\vdash$ is HSC.

Citkin's Theorem

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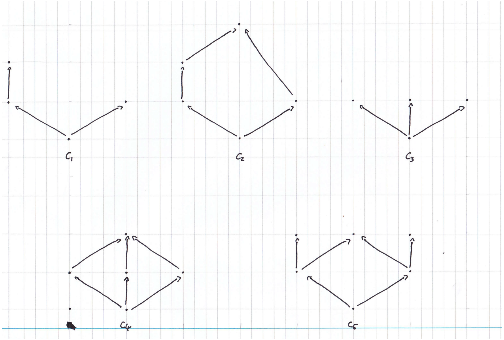
Citkin's Theorem [1978] In order for an intermediate logic Λ to be HSC it is necessary and sufficient that Λ is not included in any of the logics $Log(C_i) : 1 \leq i \leq 5$.



The Modal Case

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This strategy can also be applied to the modal case and Rybakov's result.

However, more than simply provide a new proof, this approach illuminates a mistake in Rybakov's characterisation. It is too restrictive and misses an infinite collection of HSC transitive modal logics.

We want to both correct and prove the characterisation.

The Two Characterisations

Rybakov's Theorem In order for a modal logics Λ over K4 to be HSC it is necessary and sufficient that Λ is not included in any of the logics $Log(F_i) : 1 \leq i \leq 13$.

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Revised Theorem In order for a modal logics Λ over K4 to be HSC it is necessary and sufficient that Λ is not included in any of the logics $Log(F_i) : 1 \leq i \leq 17$ and $Log(G_n)$ for some $n \in \omega$.

Proof Strategy

Characterising the HSC logics over K4 is equivalent to characterising primitive sub-vareties of K4-algebras.

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Definition A variety $\mathcal{A}$ is primitive iff every sub-quasivarity of $\mathcal{A}$ is a variety.

Theorem A normal modal logic Λ over K4 is HSC iff its corresponding variety $\mathcal{A}$ is primitive.

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Employing results from universal algebra further reduces this problem.

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Definition An algebra A is weakly projective in a variety $\mathcal{A}$ iff $\forall B \in \mathcal{A}$ iff $A \in \mathbb{H}(B)$ then $A \in \mathbb{IS}(B)$.

An algebra A is finitely subdirectly irreducible (FSI) iff the identity relation is $\wedge$-irreducible in the congruence lattice of A.

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Lemma Let $\mathcal{A}$ be a variety of K4-algebras.

(i) If $\mathcal{A}$ is primitive then the finite, non-trivial FSI members of $\mathcal{A}$ are weakly projective in $\mathcal{A}$.

(ii) Suppose all sub-vareties of $\mathcal{A}$ have the FMP. If the finite, non-trivial FSI members of $\mathcal{A}$ are weakly projective in $\mathcal{A}$ then $\mathcal{A}$ is primitive.

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Definition A transitive space is a triple $\mathcal{X} := (X, \tau, R)$ where (X, R) is a Kripke frame, (X, τ) is a Stone space and such that

(i) $R[x]$ is closed for all $x \in X$;

(ii) $R^{-1}[U]$ is clopen for all clopen $U \subseteq X$;

(iii) R is a transitive relation.

Theorem The category of K4-algebras and category of transitive spaces are dually equivalent.

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Algebra	Topology
FSI	Rooted
Sub-algebra	Quotient Space
Quotient Algebra	Closed Upset
Direct Product	Disjoint Union

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Explaining the Mistake

Theorem The variety generated by the algebraic dual of irreflexive F_3 is primitive.

Proof Sketch: The variety $\mathcal{A}$ is locally finite, so it is sufficient to show its finite non-trivial FSI members are weakly projective members are primitive.

We argue via duality that all its members have a particular shape and conclude the main result form there.

The Easy Direction

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Lemma Primitive varieties of K4-algebras omit $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$.

Proof Sketch: For each $1 \leq i \leq 17$ we show that if $\mathcal{A}$ contains $F_i^* *$ then it contains a finite, non-trivial FSI member not weakly projective in $\mathcal{A}$.

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Proof Sketch: For each $1 \leq i \leq 17$ we show that if $\mathcal{A}$ contains $F_i^* *$ then it contains a finite, non-trivial FSI member not weakly projective in $\mathcal{A}$.

For the G_n^* claim, we show that if $\mathcal{A}$ includes G_n^* for all $n \in \omega$ then it contains G_ω^* which makes G_1^* finite, non-trivial FSI but not weakly projective.

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We first establish a detailed description of the finitely generated, non-trivial SI members of the varieties.

This requires establishing a group of results demonstrating certain frame substructures never appear in our spaces.

The Difficult Direction

<u>Theorem</u> Let $\mathcal{A}$ be a variety omitting $F_i^* : 1 \leq i \leq 17$ and G_n^* for some $n > 0$. Let $A \in V$ be finitely generated, non-trivial and SI. Then the frame underlying A_* is a sequential composition of frames $\bigoplus_{\alpha \leq \beta} Q_\alpha$ for some $\beta \in Ord$ and such that:

$$Q_\alpha \text{ is } \begin{cases} \text{a single cluster} & \text{if } \alpha = \beta \text{ or } \alpha \text{ is a limit ordinal} \\ \text{a single cluster, a two cluster anti-chain or } H & \text{if } \alpha = 0 \\ \text{a single cluster or a two cluster anti-chain} & \text{otherwise} \end{cases}$$

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Moreover: Any maximal clusters are single reflexive points

If Q_α is a two cluster anti-chain then clusters in $Q_{\alpha+1}$ are improper.

If A_* contains an irreflexive point then $\beta = \lambda + n$ for some limit ordinal λ, $n \neq 0$ and $\exists 0 < m \leq n : \forall \alpha < \lambda + m \; Q_\alpha$ contains no irreflexive points, $\forall k \geq m \; Q_{\lambda+k}$ is a single irreflexive point and if $m < n$ then $Q_{\lambda+m-1}$ is a single cluster.

The Difficult Direction

Theorem All our varieties have the FMP.

Proof Sketch: We follow a variation on the drop point technique of K. Fine. Given an algebra A and formula φ it invalidates, we use the previous structural result to construct a finite sub-algebra of A that also invalidates φ.

The Difficult Direction

Theorem Let $\mathcal{A}$ be one of our varieties. Every finite, non-trivial FSI member of $\mathcal{A}$ is weakly projective in $\mathcal{A}$.

Proof Sketch: Letting $A \in \mathbb{H}(B)$ we want to show that $A \in \mathbb{IS}(B)$. By the duality this amounts to assuming A_* is a closed upset of B_* and we want to show A_* is also a p-morphic image of B_*.

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We do this by recursively collapsing B_* into A_* in a process enabled by the structural result.

Summary

Combing all our results we have a complete characterisation of primitive K4-algebras.

Theorem A variety of K4-algebras $\mathcal{A}$ is primitive iff $\mathcal{A}$ omits $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n > 0$.

Summary

Combing all our results we have a complete characterisation of primitive K4-algebras.

Theorem A variety of K4-algebras $\mathcal{A}$ is primitive iff $\mathcal{A}$ omits $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n > 0$.

Consequently we also have a complete characterisation of HSC logics over K4.

Revised Theorem In order for a modal logics Λ over K4 to be HSC it is necessary and sufficient that Λ is not included in any of the logics $Log(F_i) : 1 \leq i \leq 17$ and $Log(G_n)$ for some $n \in \omega$.

Further Study

Extend the strategy to situations with a comparable set-up. Candidates include:

1. Modal logics over wK4;
2. All modal logics;
3. Intuitionistic modal logic;
4. Multi-modal logic.

Thanks

Thank you all for listening.

Additional thanks goes to Nick and Tommaso, the supervisors of my master's thesis from which this talk is based.