Kan-injectivity and KZ-doctriness

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joint work with Ivan Di Liberti and Gabriele Lobbia

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KZ-doctrine $=$ lax-idempotent pseudomonad

$\mathbb{T} = (T, \eta, \mu, ...) : \mathcal{K} \to \mathcal{K}$ such that $T\eta \dashv \mu \dashv \eta T$

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Examples. Completions under classes of colimits

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Examples. Completions under classes of colimits

Cat [Kock, JPAA, 1995]

Lex [Garner, Lack, Lex colimits, JPAA, 2012]

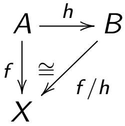
$\mathcal{V}$-Cat [Power, Cattani, Winskel, JPAA, 2000]

Pos

Top

Loc, e.g. stably locally compact locales
[Johnstone, Sketches of an Elephant]

- X is left Kan-injective w.r.t. $h : A \to B$ if there is an invertible 2-cell



exhibiting f/h as a left Kan extension of f along h;

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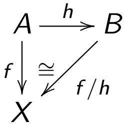

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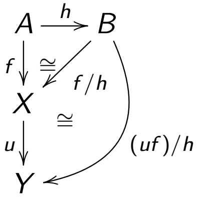

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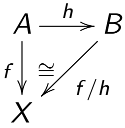

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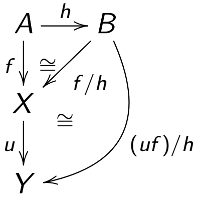

$\mathrm{LInj}(\mathcal{H})$:= locally full sub-2-category of all objects and 1-cells left Kan injective w.r.t. all $h \in \mathcal{H}$

Equivalently:

X is left Kan-injective wrt $h : A \to B$ iff

$\mathcal{K}(B,X) \xrightarrow{\mathcal{K}(h,X)} \mathcal{K}(A,X)$ is a rali:

$$\mathcal{K}(h,X)^* \xrightarrow{(\cong,\epsilon)} \mathcal{K}(h,X)$$

$u : X \to Y$ is Kan-injective wrt $h : A \to B$, iff (X and Y are so, and) it satisfies the Beck-Chevalley condition:

$$\begin{array}{ccccc} \mathcal{K}(B,X) & \xleftarrow{(\mathcal{K}(h,X))^*} & \mathcal{K}(A,X) & & X \\ {\scriptstyle \mathcal{K}(B,u)}\downarrow & \cong & \downarrow{\scriptstyle \mathcal{K}(A,u)} & & \downarrow{\scriptstyle u} \\ \mathcal{K}(B,Y) & \xleftarrow[(\mathcal{K}(h,Y))^*]{} & \mathcal{K}(A,Y) & & Y \end{array}$$

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$\longrightarrow$ [Lack, Rosicky, Enriched Weakness, 2012]

$$\mathsf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$$

creates (bi)limits and pseudolimits.

For $D : I \to \mathsf{LInj}(\mathcal{H})$, $W : I \to \mathsf{Cat}$ and $L = \mathsf{bilim}_W D$,

$$\mathcal{K}(B, L) \simeq \mathsf{Psd}[I, \mathsf{Cat}](W, \mathcal{K}(B, D-)) \underset{\perp}{\rightleftarrows} \mathsf{Psd}[I, \mathsf{Cat}](W, \mathcal{K}(A, D-)) \simeq \mathcal{K}(A, L)$$

In CAT,

$\mathrm{LInj}\big(\{D \hookrightarrow \hat{D} \mid D\,\text{small},\ \hat{D}\,\text{obtained by freely adding a terminal obj. to } D\}\big)$
$=$ cocomplete categories and functors preserving colimits

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$= \mathrm{LInj}\big(\{\, 0 \to 1,\ \boxed{a \bullet \quad \bullet b} \to \boxed{a\bullet \to \bullet \leftarrow \bullet b},\ \boxed{\bullet \rightrightarrows \bullet} \to \boxed{\bullet \rightrightarrows \bullet \to \bullet}\,\}\big)$

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In order-enriched categories, several examples are given in Pos, Top and Loc in [Adámek, S., Velebil, 2015] and [Carvalho,S., 2017]

Lax-idempotent (pseudo)monads and left Kan extensions

$\mathbb{T} = (T, \eta, \mu, ...) : \mathcal{K} \to \mathcal{K}$ lax-idempotent pseudomonad

The 2-category of (pseudo-)algebras and their homomorphisms is (up to equivalence)

$$\mathsf{LInj}\big(\{\eta_X \mid X \in \mathcal{K}\}\big)$$

$\mathsf{LInj}(\{\eta_X\}_{X\in\mathcal{K}})$ is KZ-monadic in $\mathcal{K}$.

$$\mathsf{LInj}(\{\eta_X\}_{X\in\mathcal{K}}) \underset{\hookrightarrow}{\overset{\leftarrow}{\perp}} \mathcal{K}$$

[Bunge, Funk, On a bicomma object cond. for KZ-docts., JPAA, 1999]
[Carvalho, S, Top. App., 2011] for order-enriched
[Marmolejo, Wood, Kan extensions and lax idemp. pseudomonads, TAC, 2012]

lax-idempotent pseudomonad	idempotent monad
$\mathsf{LInj}\big(\{\eta_X \mid X \in \mathcal{K}\}\big)$	$\mathsf{Orth}\big(\{\eta_X \mid X \in \mathcal{K}\}\big)$
Kan Injectivity Subcategory Problem	Orthogonal Subcategory Problem When is $\mathsf{Orth}(\mathcal{H})$ a full reflective subcategory?

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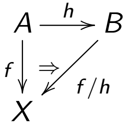
When is $\mathsf{LInj}(\mathcal{H})$ KZ-monadic?

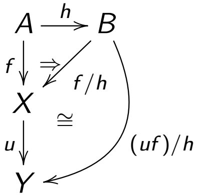
Answered for order-enriched categories in
[Adámek, S., Velebil, Kan-injectivity in order-enrich. cats., MSCS, 2015]

- X is weak left Kan-injective w.r.t. $h : A \to B$ if there is a 2-cell



exhibiting f/h as a left Kan extension of f along h;

- $X \xrightarrow{u} Y$ is weak left Kan-injective w.r.t. $h : A \to B$ if Z and X are so and u preserves left Kan extensions along h:



WLInj($\mathcal{H}$) := locally full sub-2-category of all objects and 1-cells weak left Kan injective w.r.t. all $h \in \mathcal{H}$

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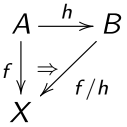

see, for instance, [Bunge, Funk, JPAA,1999]

exhibiting f/h as a left Kan extension of f along h;

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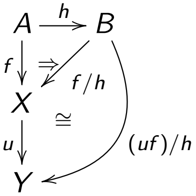


WLInj($\mathcal{H}$) := locally full sub-2-category of all objects and 1-cells weak left Kan injective w.r.t. all $h \in \mathcal{H}$

Example

In Cat,

$\mathsf{Rex} = \mathsf{WLInj}(\{D \to 1 \mid D \text{ finite}\})$ [MacLane, Categories...]

$= \mathsf{WLInj}(\{\, 0 \to 1,\ \boxed{a\bullet \quad \bullet b} \to 1,\ \boxed{\bullet \rightrightarrows \bullet} \to 1 \,\})$

$\mathsf{Rex} = \mathsf{LInj}(\{D \hookrightarrow \hat{D} \mid D \text{ finite}\})$ [Riehl, C.T. in Context]

In a 2-category with (bi-)cocomma objects,

for every $\mathcal{H}$ there is $\overline{\mathcal{H}}$ with

$$\mathsf{WLInj}(\mathcal{H}) = \mathsf{LInj}(\overline{\mathcal{H}})$$

$$\begin{array}{ccc} A & \xrightarrow{h \in \mathcal{H}} & A' \\ {\scriptstyle 1_A}\downarrow & \Longrightarrow & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i_h \in \overline{\mathcal{H}}]{} & A_h \end{array}$$

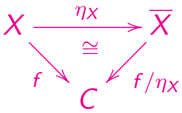
When is $\mathsf{LInj}(\mathcal{H})$ KZ-monadic?

Let $\mathcal{C}$ be a locally full (locally replete) sub-2-cat. of $\mathcal{K}$, let

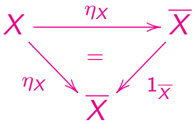
$$X \xrightarrow{\eta_X} \overline{X}\ , \quad \text{with } \overline{X} \in \mathcal{C} \qquad (X \in \mathcal{K})$$

be such that

1. $\mathcal{C} \subseteq \mathrm{LInj}(\{\eta_X\}_{X \in \mathcal{K}})$
2. $f/\eta_X \in \mathcal{C}$, for every $X \xrightarrow{f} C$, with $C \in \mathcal{C}$



3. every η_X is dense, i.e., $\eta_X/\eta_X \cong 1_{\overline{X}}$



Then there is a KZ-adjunction with the right part given by the inclusion

$$\mathcal{C} \hookrightarrow \mathcal{K}$$

For order enriched categories, in [Carvalho, S., TA, 2011]

For the general context, it follows easily from [Marmolejo, Wood, TAC, 2012]

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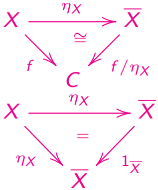

Then there is a KZ-adjunction with the right adjoint given by the inclusion

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Moreover, if $\mathcal{C} = \mathrm{LInj}(\mathcal{H})$, for some $\mathcal{H}$, then

$\mathrm{LInj}(\mathcal{H})$ is KZ-monadic

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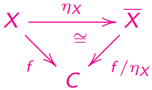

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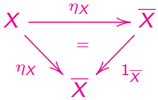

Then there is a KZ-adjunction with the right adjoint given by the inclusion

$$\mathcal{C} \hookrightarrow \mathcal{K}$$

$$\mathrm{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$$

A transfinite construction of η_X:

$$X = \underbrace{X_0 \to X_1 \to X_2 \to \cdots \to X_k}_{\eta_X} \to \ldots X_i \to \ldots \qquad (i \in \mathrm{Ord})$$

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- for KISP in order-enriched: colimits of chains, pushs., mult. pushouts and coinserters [Adámek, S., Velebil, 2015]

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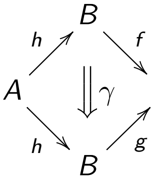
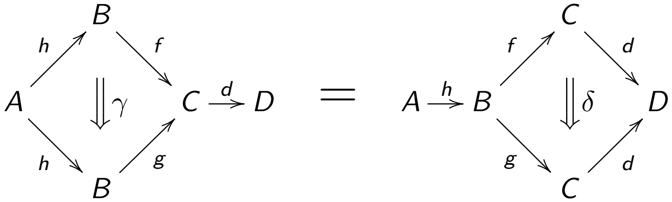
Present context: (bi-)colimits of (pseudo-)chains, (bi-)pushouts, (bi-)wide-pushouts and (bi-)coequinserters

The (bi-)coequinserter of a 2-cell



consists of

a 1-cell $d : C \to D$ and a 2-cell $\delta : df \Rightarrow dg$ such that

- 

- universal conditions

With $(c, \alpha) = coins(f, g)$ and $q = coequifier(c\gamma, \alpha h)$, $d = qc$ and $\delta = d\alpha$.

$\longrightarrow$ We could work just with 2-colimits, but bicolimits are more realistic.

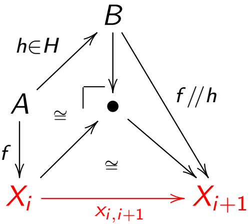
For instance, Lex has bicolimits but not 2-colimits.

The (pseudo)chain $X = X_0 \xrightarrow{x_{01}} X_1 \to \cdots \to X_i \xrightarrow{x_{ij}} X_j \ldots$

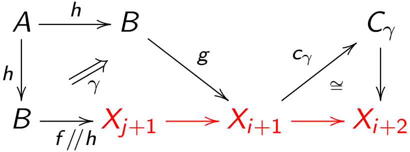
Limit steps. $X_i = (bi)colim(x_{ji})_{j<i}$

Isolated steps:

i+1:



i+2:



$j \leq i, j$ even

For every ordinal i,

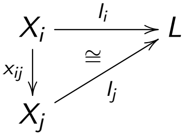
$\mathrm{LInj}(\mathcal{H})$ is left Kan injective with respect to

$$X = X_0 \xrightarrow{x_{0i}} X_i$$

An object A is λ-small if

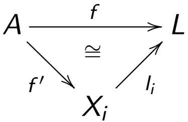
$\mathcal{K}(A, -) : \mathcal{K} \to \mathrm{Cat}$ preserves bicolimits of λ-pseudo-chains:

Given a bicolimit of a λ-pseudo-chain

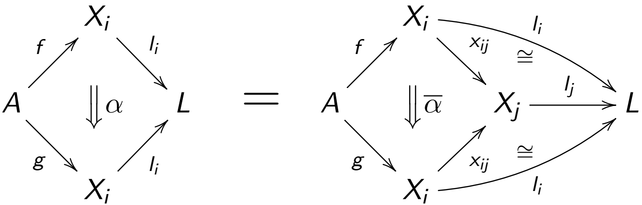


$(i \leq j,\ i, j \in \lambda)$

1. For every $A \xrightarrow{f} L$, there is f' and $i \in \lambda$ with



2. For every 2-cell $\alpha : l_i f \Rightarrow l_i g$, there is $j \geq i$ and $\overline{\alpha} : x_{ij} f \Rightarrow x_{ij} g$ with



Let $\mathcal{K}$ be a 2-category with (bi-)colimits where every object A is κ_A-small for some κ_A.

Then, for every set $\mathcal{H}$ of 1-cells, the inclusion

$$\mathsf{LInj}(\mathcal{H}) \hookrightarrow \mathcal{K}$$

is the right part of a KZ-adjunction.

Moreover, $\mathsf{LInj}(\mathcal{H})$ is KZ-monadic, that is, it is, up to equivalence, the 2-category of (pseudo)algebras of the corresponding lax idempotent pseudomonad.

Take $\kappa \geq \kappa_A$, for A domain or codomain of some $h \in \mathcal{H}$.

$X_0 \xrightarrow{x_{0\kappa}} X_\kappa$ plays the role of $X \xrightarrow{\eta_X} \overline{X}$.

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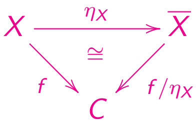
- $\mathsf{LInj}(\mathcal{H}) \subseteq \mathsf{LInj}(\{\eta_X\}_{X\in\mathcal{K}})$
- $f/\eta_X \in \mathsf{LInj}(\mathcal{H})$, for every $X \xrightarrow{f} C$, with $C \in \mathsf{LInj}(\mathcal{H})$



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