Two-layered Belnapian logics for uncertainty

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Motivation: Belief based on information

- It is natural to view belief as based on evidence/information
- Potential incompleteness, uncertainty, and contradictoriness of information needs to be dealt with adequately
- Separately, these characteristics has been taken into account by various appropriate logical formalisms and (classical) probability theory
- The first two are often accommodated within one formalism (e.g. imprecise probability), the second two less so.
- Conflict or contradictoriness of information is rather to be *resolved* than to be *reasoned with*.

Two-dimensionality of information

Addressing incompleteness and contradictoriness of information in one framework:

- separating positive and negative information, which are not considered complementary and can overlap
- semantically, distinguishing support for from opposition to a statement (or qualifying/quantifying evidence for and evidence against a statement being the case separately)
- explicit in the double-valuation semantics of Belnap-Dunn logic, and the concept of bi-lattices or twist product algebras.
- this approach can be extended to encompass uncertainty measures like *probabilities*, *belief functions*, and *graded reasoning*.

Two-layer logics for uncertainty

Two-layer syntax. $(\mathcal{L}_i, \mathcal{M}, \mathcal{L}_o)$ with

- inner language $\mathcal{L}_i$ *(events, evidence)* — CPC
- outer language $\mathcal{L}_o$ *(agent, belief)* — Ł or linear inequalities
- $\mathcal{M}$: Modalities $m : (\mathcal{L}_i)^n \to \mathcal{L}_o$ — P

Two-layer semantics consists of

– semantics of $\mathcal{L}_i$ — $(\mathcal{P}(W), \cap, \cup, -)$
– interpretation of modalities $\mathcal{M}$ — $\mu : \mathcal{P}(W) \to [0,1]$
– semantics of $\mathcal{L}_o$ — $[0,1]_{Ł}$

Two-layer axiomatization of $L = (L_i, M, L_o)$ consists of

– a complete axiomatics of the inner logic L_i
– modal axioms and rules M — $P\neg\varphi \leftrightarrow {\sim}P\varphi$ or $P\neg\varphi = 1 - P\varphi$
– a complete axiomatics of the outer logic L_o.

Two-layer logics for uncertainty

- **Fagin, Halpern, Meggido 1990's**: two-layer logics for reasoning about probability and belief
(CPC, probability, reasoning about linear inequalities),
- **Zhou 2013**: generalization of belief functions, and the logics above, to distributive lattices
(BD, belief, reasoning about linear inequalities),
- **Hájek, Godo, Esteva 1995**: two-layer modal logics, with many-valued modality "probably" (CPC, P, Ł),
- **Cintula & Noguera 2014**: an abstract framework of two-layer modal logics, with a general theory of syntax, semantics and completeness.
- **This talk**: two-layer modal logics, with a many-valued modalities based on Belnapian probabilities or belief (and plausibility) functions
(BD, M, L_o) with L_o derived from Łukasiewicz logic.

Belnapian two-layer logics for uncertainty

Two-layer syntax. $(\mathcal{L}_i, \mathcal{M}, \mathcal{L}_o)$ with

- inner language $\mathcal{L}_i$ *(evidence)* BD
- outer language $\mathcal{L}_o$ *(agent, belief)* $Ł_\neg$ or linear inequalities
- $\mathcal{M}$: Modalities $m : (\mathcal{L}_i)^n \to \mathcal{L}_o$ P

Two-layer semantics consists of

– semantics of $\mathcal{L}_i$ $(\mathcal{P}(S)^{\bowtie}, \wedge, \vee, \neg)$

– interpretation of modalities $\mathcal{M}$ $\mu : \mathcal{P}(S) \to [0, 1]$

– semantics of $\mathcal{L}_o$ $[0, 1]^{\bowtie}_{Ł}$

Two-layer axiomatization of $L = (L_i, M, L_o)$ consists of

– a complete axiomatics of the inner logic L_i

– modal axioms and rules M $P\neg\varphi \leftrightarrow \neg P\varphi$ or $P^+\neg\varphi = P^-\varphi$

– a complete axiomatics of the outer logic L_o.

Belnap-Dunn as the inner logic: qualifying evidence

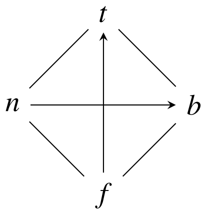
Language $\mathcal{L}_{\mathsf{BD}}$: $\varphi := p \in \mathrm{Prop} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi$

$(\mathbf{4}, \wedge, \vee, \neg)$ is a de Morgan algebra

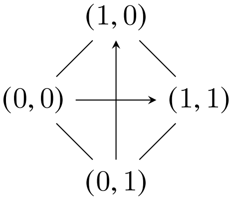
- $(\mathbf{4}, \wedge, \vee)$ is a distributive lattice
- each element represents the availability of positive and/or negative information
  - t: true (top)
  - n: no info b: contradictory info
  - f: false (bottom)
- $\neg$ is an involutive de Morgan negation.

BD consequence relation and Exactly true logic

$\Gamma \vDash_{\mathsf{BD}} \varphi$ given as preservation of $\{t, b\}$.
$\Gamma \vDash_{\mathsf{ETL}} \varphi$ given as preservation of $\{t\}$.



Belnap-Dunn square $\mathbf{4}$



Belnap-Dunn as the inner logic: axiomatics

BD is completely axiomatized using the following axioms and rules:

$$\varphi \wedge \psi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi \qquad \varphi \vdash \psi \vee \varphi \qquad \varphi \vdash \varphi \vee \psi$$

$$\varphi \vdash \neg\neg\varphi \qquad \neg\neg\varphi \vdash \varphi \qquad \varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

$$\neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi) \qquad \neg(\varphi \wedge \psi) \vdash \neg\varphi \vee \neg\psi$$

$$\frac{\varphi \vdash \psi,\ \psi \vdash \chi}{\varphi \vdash \chi} \qquad \frac{\varphi \vdash \psi,\ \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi} \qquad \frac{\varphi \vdash \chi,\ \psi \vdash \chi}{\varphi \vee \psi \vdash \chi} \qquad \frac{\varphi \vdash \psi}{\neg\psi \vdash \neg\varphi}$$

- $\Gamma \vdash_{\mathsf{BD}} \varphi$ is the consequence relation generated by the above
- BD is strongly complete and locally finite.
- BD allows for a unique (irredundant) DNF and CNF.
- ETL: $\vdash_{\mathsf{ETL}}$ is obtained from $\vdash_{\mathsf{BD}}$ adding $\neg\varphi \wedge (\varphi \vee \psi) \vdash \psi$

Belnap-Dunn as the inner logic: frame semantics [Dunn 76]

Language $\mathcal{L}_{\mathrm{BD}}$ $\varphi := p \in \mathrm{Prop} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi$

4-valued Models $M = \langle S,\ v : S \times \mathrm{Prop} \to \mathbf{4} \rangle$
v is extended to formulas in the standard way.

Double-valuation semantics $M = \langle S, \Vdash^{+}, \Vdash^{-} \rangle$

$s \Vdash^{+} \varphi$ iff $v(s)(\varphi) \in \{t, b\}$ $\qquad$ $s \Vdash^{-} \varphi$ iff $v(s)(\varphi) \in \{f, b\}$.

i.e.

$s \Vdash^{+} \varphi \wedge \psi$ iff $s \Vdash^{+} \varphi$ and $s \Vdash^{+} \psi$ $\qquad$ $s \Vdash^{-} \varphi \wedge \psi$ iff $s \Vdash^{-} \varphi$ or $s \Vdash^{-} \psi$

$s \Vdash^{-} \varphi \vee \psi$ iff $s \Vdash^{-} \varphi$ and $s \Vdash^{-} \psi$ $\qquad$ $s \Vdash^{+} \varphi \vee \psi$ iff $s \Vdash^{+} \varphi$ or $s \Vdash^{+} \psi$

$s \Vdash^{+} \neg\varphi$ iff $s \Vdash^{-} \varphi$ $\qquad$ $s \Vdash^{-} \neg\varphi$ iff $s \Vdash^{+} \varphi$

$|\varphi|^{+} = \{s \mid s \Vdash^{+} \varphi\}$ $\qquad$ $|\varphi|^{-} = \{s \mid s \Vdash^{-} \varphi\}$

Consequence relation $\Gamma \vDash_{\mathrm{BD}} \varphi$ iff $\forall M, s\ (s \Vdash^{+} \Gamma \to s \Vdash^{+} \varphi)$.

Belnapian probabilities: quantifying evidence

- $\mathtt{m} : S \to [0,1]$ a mass function: $\sum_{s \in S} \mathtt{m}(s) = 1$
- $\mathrm{p} : \mathcal{P}S \to [0,1]$ given by

$$\mathrm{p}(X) = \sum \{\mathtt{m}(s) \mid s \in X\}$$

- Generates an assignment $(\mathrm{p}^+, \mathrm{p}^-) : L_{\mathrm{BD}} \to [0,1] \times [0,1]^{op}$:

$$\mathrm{p}^+(\varphi) = \mathrm{p}(|\varphi|^+) = \sum \left\{\mathtt{m}(s) \mid s \Vdash^+ \varphi\right\}$$

$$\mathrm{p}^-(\varphi) = \mathrm{p}(|\varphi|^-) = \mathrm{p}^+(\neg\varphi) \quad \neg\text{-coherence}$$

The probability function p^+ satisfies:

(A1) normalization $0 \leq \mathrm{p}^+(\varphi) \leq 1$
(A2) monotonicity if $\varphi \vdash_{\mathrm{BD}} \psi$ then $\mathrm{p}^+(\varphi) \leq \mathrm{p}^+(\psi)$
(A3) incl.-excl. $\mathrm{p}^+(\varphi \wedge \psi) + \mathrm{p}^+(\varphi \vee \psi) = \mathrm{p}^+(\varphi) + \mathrm{p}^+(\psi)$.

- D. Klein, O. Majer, S. Raffie-Rad, *Probabilities with gaps and gluts*, JPL 2021.
- C. Zhou, Belief functions on distributive lattices. Artif. Intell. 201, (2013).

Belnapian probabilities: quantifying evidence

- $\mathtt{m} : \mathcal{P}\mathtt{Lit} \to [0,1]$ a mass function: $\sum_{\Gamma \subseteq \mathtt{Lit}} \mathtt{m}(\Gamma) = 1$
- Generates an assignment $(\mathrm{p}^+, \mathrm{p}^-) : L_{\mathsf{BD}} \to [0,1] \times [0,1]^{op}$:

$$\mathrm{p}^+(\varphi) = \sum \{\mathtt{m}(\Gamma) \mid \Gamma \vdash \varphi\}$$

$$\mathrm{p}^-(\varphi) = \mathrm{p}^+(\neg\varphi) \quad \neg\text{-coherence}$$

The probability function p^+ satisfies:

(A1) normalization	$0 \leq \mathrm{p}^+(\varphi) \leq 1$
(A2) monotonicity	if $\varphi \vdash_{\mathsf{BD}} \psi$ then $\mathrm{p}^+(\varphi) \leq \mathrm{p}^+(\psi)$
(A3) incl.-excl.	$\mathrm{p}^+(\varphi \wedge \psi) + \mathrm{p}^+(\varphi \vee \psi) = \mathrm{p}^+(\varphi) + \mathrm{p}^+(\psi)$.

- D. Klein, O. Majer, S. Raffie-Rad, *Probabilities with gaps and gluts*, JPL 2021.
- C. Zhou, Belief functions on distributive lattices. Artif. Intell. 201, (2013).

Other uncertainty measures $\mathtt{p}, \mathtt{bel}, \mathtt{pl} : \mathcal{P}S \to [0,1]$

(A1) normalization, (A2) monotonicity, and

Inner probabilities $(\mathtt{p}^+, \mathtt{p}^-) : \mathtt{p}^-(\varphi) = \mathtt{p}^+(\neg\varphi)$

(A3) incl.-excl. $\mathtt{p}^+(\varphi \vee \psi) \geq \mathtt{p}^+(\varphi) + \mathtt{p}^+(\psi) - \mathtt{p}^+(\varphi \wedge \psi)$

General belief functions $(\mathtt{bel}^+, \mathtt{bel}^-) : \mathtt{bel}^-(\varphi) = \mathtt{bel}^+(\neg\varphi)$

(A_n) n-monotonicity $$\mathtt{bel}^+(\bigvee_{i=1}^{n} \varphi_i) \geq \sum_{\substack{J \subseteq \{1,\dots,n\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot \mathtt{bel}^+\left(\bigwedge_{j \in J} \varphi_j\right)$$

General plausibility functions $(\mathtt{pl}^+, \mathtt{pl}^-) : \mathtt{pl}^-(\varphi) = \mathtt{pl}^+(\neg\varphi)$

(A_n) n-monotonicity $$\mathtt{pl}^+(\bigwedge_{i=1}^{n} \varphi_i) \leq \sum_{\substack{J \subseteq \{1,\dots,n\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot \mathtt{pl}^+\left(\bigvee_{j \in J} \varphi_j\right)$$

Similarly, we can consider possibilities and necessities, or qualitative probabilities.

Example: belief based on (multiple) information sources

- A model provides sets $|\varphi|^+ = \{s \mid s \Vdash^+ \varphi\}$ and $|\varphi|^- = \{s \mid s \Vdash^- \varphi\}$
- They can intersect and do not have to cover S
- each **source** provides probabilities $(\mathtt{p}^+(\varphi), \mathtt{p}^-(\varphi))$ (or belief functions $(\mathtt{bel}^+(\varphi), \mathtt{bel}^-(\varphi))$)
- an aggregation provides an assignment $(B^+(\varphi), B^-(\varphi))$ = a degree of **belief**

States of a model:

$\Vdash^+ \varphi$	$\Vdash^+ \varphi$ $\Vdash^- \varphi$
	$\Vdash^- \varphi$

- Belief assignment can be a Belnapian probability: then it satisfies the **probability axioms**,
- it can be a Belnapian belief function: then it satisfies the **belief function axioms**,
- or, it can be just **monotone** and **coherent**:

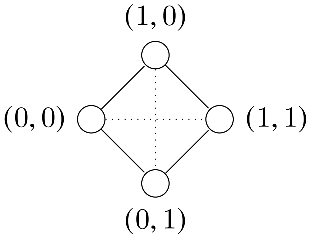
$$\varphi \vdash_{\mathsf{BD}} \psi \;/\; B^+(\varphi) \leq B^+(\psi) \qquad B^-(\varphi) = B^+(\neg\varphi).$$

Belnapian uncertainty measures: the range $[0,1] \times [0,1]^{op}$

Continuous extension of $\mathbf{4}$:
the twist product $[0,1]^{\bowtie}$ with $\mathbf{L}_{[0,1]} = ([0,1], \min, \max)$.

The twist product $[0,1]^{\bowtie}$

$$(a_1, a_2) \wedge (b_1, b_2) = (a_1 \wedge b_1, a_2 \vee b_2)$$
$$(a_1, a_2) \vee (b_1, b_2) = (a_1 \vee b_1, a_2 \wedge b_2)$$
$$\neg(a_1, a_2) = (a_2, a_1)$$



- $(\mathrm{p}^+(\varphi), \mathrm{p}^-(\varphi))$: positive/negative probabilistic support of φ
- “classical” vertical line: $\mathrm{p}^+(\varphi) = 1 - \mathrm{p}^-(\varphi)$,
- Graded reasoning about (belief based on) probabilities or belief functions can be interpreted over expansions of $[0,1]^{\bowtie}$.

Two-dimensional outer logics for probabilities and belief functions

- to be interpreted over an algebra (matrix) expanding $[0,1]^{\bowtie}$ with implication, fusion, negation, ...
- to be able to express all three probability (belief functions) axioms – derived from Łukasiewicz logic and $[0,1]_{Ł}$
- two ways of negating implication
  (a) "de Morgan" way, using a co-implication

$$\neg(a \to b) := (\neg b \ominus \neg a)$$

  (b) "Nelson" way, combining positive and negative semantical values

$$\neg(a \rightharpoonup b) := (a \,\&\, \neg b)$$

Two-dimensional logics for comparative uncertainty

- to be interpreted over an algebra (matrix) expanding $[0,1]^{\bowtie}$ with implication, fusion, negation, ...
- to be able to express: monotonicity and coherence in case of comparative uncertainty
  – derived from Gödel logic and $[0,1]_G$
- two ways of negating implication
  (a) "de Morgan" way, using a co-implication (bi-Gödel logic)

$$\neg(a \to b) := (\neg b \prec \neg a)$$

  (b) "Nelson" way, combining positive and negative semantical values

$$\neg(a \rightharpoonup b) := (a \ \wedge \ \neg b)$$

case (a): Ł^2, reasoning with probabilities or bel. functions

Standard MV algebra

$[0,1]_{Ł} = ([0,1], \wedge, \vee, \&_{Ł}, \rightarrow_{Ł})$:

$$a \wedge b := \min\{a, b\}, \qquad a \&_{Ł} b := \max\{0, a+b-1\}$$
$$a \vee b := \max\{a, b\} \qquad a \rightarrow_{Ł} b := \min\{1, 1-a+b)\}$$
$$\sim_{Ł} a := a \rightarrow_{Ł} 0 = 1-a \qquad a \&_{Ł} b \leq c \Leftrightarrow b \leq a \rightarrow_{Ł} c$$

Definable connectives:

$$a \oplus_{Ł} b := \sim a \rightarrow_{Ł} b = \min\{1, a+b\}$$
$$a \ominus_{Ł} b := \sim(a \rightarrow_{Ł} b) = \max\{0, a-b\}$$
$$c \leq a \oplus_{Ł} b \Leftrightarrow c \ominus_{Ł} b \leq a$$

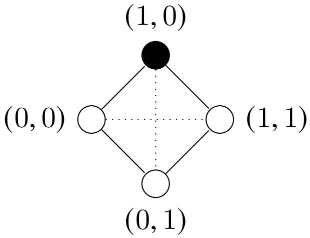
$\ominus_{Ł}$ is a co-implication.

case (a): $Ł^2$, reasoning with probabilities or bel. functions

twist product $[0,1]^{\bowtie}$, $\neg(a_1,a_2) = (a_2,a_1)$, $F = \{(1,0)\}$,

$[0,1]^{\bowtie}$ expanded with

$$(a_1,a_2) \to (b_1,b_2) = (a_1 \to_Ł b_1, b_2 \ominus_Ł a_2)$$
$$(a_1,a_2) \& (b_1,b_2) = (a_1 \&_Ł b_1, a_2 \oplus_Ł b_2)$$
$$\sim(a_1,a_2) = (\sim_Ł a_1, \sim_Ł a_2)$$



Notice: $\neg$ is symmetry along the horizontal, $\sim$ is symmetry along the middle point, $\sim\neg$ is symmetry along the vertical (conflation).
$\neg\alpha \leftrightarrow \sim\alpha$ defines the vertical. $\neg$ and $\sim$ are distinct.

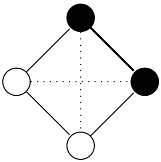
$\Gamma \models_{Ł^2} \alpha$ defined as preservation of $(1,0)$.
Its $(\wedge, \vee, \neg)$-fragment coincides with ETL.

case (a): $Ł^2$, reasoning with probabilities or bel. functions

twist product $[0,1]^{\bowtie}$, $\neg(a_1,a_2) = (a_2,a_1)$, $F = (1,1)^{\uparrow}$,

$[0,1]^{\bowtie}$ expanded with

$$(a_1,a_2) \rightarrow (b_1,b_2) = (a_1 \rightarrow_{Ł} b_1, b_2 \ominus_{Ł} a_2)$$
$$(a_1,a_2) \& (b_1,b_2) = (a_1 \&_{Ł} b_1, a_2 \oplus_{Ł} b_2)$$
$$\sim(a_1,a_2) = (\sim_{Ł} a_1, \sim_{Ł} a_2)$$



Notice: $\neg$ is symmetry along the horizontal, $\sim$ is symmetry along the middle point, $\sim\neg$ is symmetry along the vertical (conflation).
$\neg\alpha \leftrightarrow \sim\alpha$ defines the vertical. $\neg$ and $\sim$ are distinct.

$\Gamma \models_{Ł^2_{(1,1)^{\uparrow}}} \alpha$ defined as preservation of $(1,1)^{\uparrow}$.
Its $(\wedge, \vee, \neg)$-fragment coincides with BD.

case (a): Ł^2, reasoning with probabilities or bel. functions

Ł^2: Ł **expanded with the bi-lattice negation ¬.**

Axiomatization of Ł^2

$$\alpha \to (\beta \to \alpha)$$
$$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$$
$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$$
$$(\sim\beta \to \sim\alpha) \to (\alpha \to \beta)$$

$$\neg\neg\alpha \leftrightarrow \alpha$$
$$\neg\sim\alpha \leftrightarrow \sim\neg\alpha$$
$$(\sim\neg\alpha \to \sim\neg\beta) \leftrightarrow \sim\neg(\alpha \to \beta)$$
$$\alpha, \alpha \to \beta \vdash \beta \quad \alpha \vdash \sim\neg\alpha$$

- ¬- negation normal form
- Local Deduction Theorem:

$$\Gamma, \alpha \vdash_{Ł^2} \beta \text{ iff } \exists n\, \Gamma \vdash_{Ł^2} (\sim\neg\alpha)^n \to \beta$$

Theorem (FSSC):

Ł^2 is finitely strongly standard–complete w.r.t. $(([0,1]^{\bowtie}, \to, \sim), \{(1,0)\})$.

case (a): $Ł^2_{(1,1)^{\uparrow}}$

Axiomatization of $Ł^2_{(1,1)^{\uparrow}}$

$$\alpha \to (\beta \to \alpha) \qquad \neg\neg\alpha \leftrightarrow \alpha$$
$$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)) \qquad \neg{\sim}\alpha \leftrightarrow {\sim}\neg\alpha$$
$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha) \qquad ({\sim}\neg\alpha \to {\sim}\neg\beta) \leftrightarrow {\sim}\neg(\alpha \to \beta)$$
$$({\sim}\beta \to {\sim}\alpha) \to (\alpha \to \beta) \qquad \alpha, \alpha \to \beta \vdash \beta \quad \vdash \alpha / \vdash {\sim}\neg\alpha$$

- $\neg$- negation normal form
- Local Deduction Theorem:

$$\Gamma, \alpha \vdash_{Ł^2_{(1,1)^{\uparrow}}} \beta \text{ iff } \exists n\, \Gamma \vdash_{Ł^2_{(1,1)^{\uparrow}}} \alpha^n \to \beta$$

Theorem (FSSC):

$Ł^2_{(1,1)^{\uparrow}}$ is finitely strongly standard–complete w.r.t. $(([0,1]^{\bowtie}, \to, \sim), (1,1)^{\uparrow})$.

Adding a $\triangle$ operator: $Ł^2\triangle$

On the standard MV algebra: $\triangle_Ł a = \begin{cases} 1, & \text{if } a = 1 \\ 0 & \text{else} \end{cases}$

($\triangle$1)	$\triangle\alpha \vee {\sim}\triangle\alpha$
($\triangle$2)	$\triangle\alpha \to \alpha$
($\triangle$3)	$\triangle\alpha \to \triangle\triangle\alpha$
($\triangle$4)	$\triangle(\alpha \vee \beta) \to \triangle\alpha \vee \triangle\beta$
($\triangle$5)	$\triangle(\alpha \to \beta) \to \triangle\alpha \to \triangle\beta$
($\triangle$6)	$\neg\triangle\alpha \leftrightarrow {\sim}\triangle{\sim}\neg\alpha$
(Nec)	$\alpha / \triangle\alpha$

$$\triangle(a_1, a_2) = (\triangle_Ł a_1, {\sim}_Ł\triangle_Ł{\sim}_Ł a_2)$$

Globalization operator on $[0,1]_Ł^{\bowtie}$: $\Delta\alpha := \triangle\alpha \wedge {\sim}\neg\triangle\alpha$

$$\Delta(a_1, a_2) = \begin{cases} (1,0), & \text{if } (a_1, a_2) = (1,0) \\ (0,1) & \text{else} \end{cases}$$

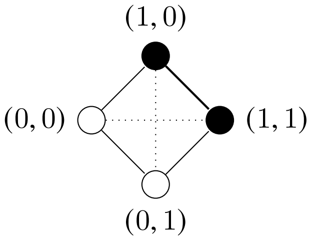
- $\neg$- negation normal form
- Δ-Deduction Theorem: $\Gamma, \alpha \vdash_{Ł^2\triangle} \beta$ iff $\Gamma \vdash_{Ł^2\triangle} \Delta\alpha \to \beta$
- Finite strong standard completeness (FSSC)

Case (b): NŁ, reasoning with probabilities or bel. functions

twist product $[0,1]^{\bowtie}$, $\neg(a_1,a_2) = (a_2,a_1)$, $F = (1,1)^{\uparrow}$:

$[0,1]^{\bowtie}$ expanded with

$$(a_1,a_2) \rightharpoonup (b_1,b_2) = (a_1 \to_{Ł} b_1, a_1 \&_{Ł} b_2)$$
$$(a_1,a_2)\&(b_1,b_2) = (a_1 \&_{Ł} b_1, a_1 \to_{Ł} \sim_{Ł} b_1)$$
$$\sim(a_1,a_2) = (\sim_{Ł} a_1, a_1)$$



$\Gamma \models_{NŁ} \alpha$ defined as preservation of $F = \{(1,a) \mid a \in [0,1]\}$.
Its $(\wedge, \vee, \neg)$-fragment coincides with BD.

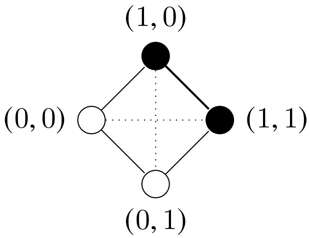
The weak equivalence $\alpha \leftrightharpoons \beta := (\alpha \rightharpoonup \beta) \wedge (\beta \rightharpoonup \alpha)$ is not congruential, the strong one $\alpha \Longleftrightarrow \beta := (\alpha \leftrightharpoons \beta) \wedge (\neg\alpha \leftrightharpoons \neg\beta)$ is.

Case (b): NŁ, reasoning with probabilities or bel. functions

twist product $[0,1]^{\bowtie}$, $\neg(a_1,a_2) = (a_2,a_1)$, $F = (1,1)^{\uparrow}$:

$[0,1]^{\bowtie}$ expanded with

$$(a_1,a_2) \rightarrowtriangle (b_1,b_2) = (a_1 \rightarrow_{Ł} b_1, a_1 \&_{Ł} b_2)$$
$$(a_1,a_2) \& (b_1,b_2) = (a_1 \&_{Ł} b_1, a_1 \rightarrow_{Ł} {\sim_{Ł}} b_1)$$
$${\sim}(a_1,a_2) = ({\sim_{Ł}} a_1, a_1)$$



$\Gamma \vDash_{NŁ} \alpha$ defined as preservation of $F = \{(1,a) \mid a \in [0,1]\}$.
Its $(\wedge, \vee, \neg)$-fragment coincides with BD.

${\sim}\alpha$ is always on the vertical. ${\sim}\alpha \leftrightarrowtriangle \neg\alpha$ defines the vertical, ${\sim}\alpha \rightarrowtriangle \neg\alpha$ defines the right triangle, and $\neg\alpha \rightarrowtriangle {\sim}\alpha$ the left. $(\alpha \rightarrowtriangle \beta) \wedge (\neg\alpha \rightarrowtriangle \neg\beta)$ captures the information order.

Case (b): NŁ, reasoning with probabilities or bel. functions

Axiomatics of NŁ:

The axioms of Łukasiewicz logic (in terms of $\rightarrowtriangle$) with MP as the only rule, plus the ¬-axioms:

$$\neg\neg\alpha \leftrightarrowtriangle \alpha$$
$$\neg(\alpha \wedge \beta) \leftrightarrowtriangle \neg\alpha \vee \neg\beta$$
$$\neg(\alpha \vee \beta) \leftrightarrowtriangle \neg\alpha \wedge \neg\beta$$
$$\neg(\alpha \rightarrow \beta) \leftrightarrowtriangle (\alpha \& \neg\beta)$$
$$\neg(\alpha \& \beta) \leftrightarrowtriangle (\alpha \rightarrow {\sim}\beta)$$
$$\neg{\sim}\alpha \leftrightarrowtriangle \alpha$$

- ¬-negation normal form (weakly equivalent only)
- Local Deduction Theorem as in Ł
- Finite strong standard completeness (FSSC)

case II.(a): $G^2_{(1,0)}(\rightarrow)$, comparative uncertainty

Standard Gödel algebra:

$[0,1]_G = ([0,1], \wedge, \vee, \rightarrow_G)$

$$a \rightarrow_G b = \begin{cases} 1, & \text{if } a \leq b \\ b & \text{else} \end{cases} \qquad {\sim_G}\, a := a \rightarrow_G 0$$

$$c \leq a \rightarrow_G b \text{ iff } a \wedge c \leq b$$

can be expanded by a co-implication:

$$b \prec_G a = \begin{cases} 0, & \text{if } b \leq a \\ b & \text{else} \end{cases} \qquad -_G a := 1 \prec_G a$$

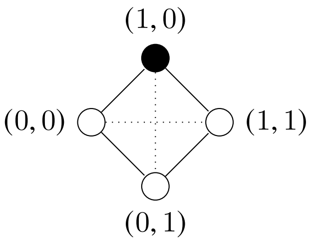
$$b \prec_G a \leq c \text{ iff } b \leq a \vee c$$

case (a): $G^2_{(1,0)}(\rightarrow)$, comparative uncertainty

twist product $[0,1]^{\bowtie}$, $\neg(a_1,a_2) = (a_2,a_1)$, $F = \{(1,0)\}$,

$[0,1]^{\bowtie}$ expanded with

$$(a_1,a_2) \rightarrow (b_1,b_2) = (a_1 \rightarrow_G b_1, b_2 \prec_G a_2)$$
$$(a_1,a_2) \prec (b_1,b_2) = (a_1 \prec_G b_1, b_2 \rightarrow_G a_2)$$
$$\sim(a_1,a_2) = (\sim_G a_1, -_G a_2)$$



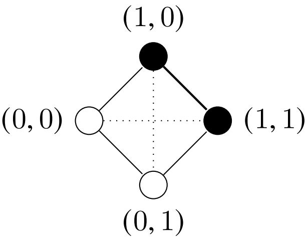
$\Gamma \vDash_{G^2_{(1,0)}(\rightarrow)} \alpha$ defined as preservation of $(1,0)$.
Its $(\wedge, \vee, \neg)$-fragment coincides with ETL.

case (a): $G^2_{(1,1)^\uparrow}(\to)$, comparative uncertainty

twist product $[0,1]^{\bowtie}$, $\neg(a_1,a_2) = (a_2,a_1)$, $F = (1,1)^\uparrow$,

$[0,1]^{\bowtie}$ expanded with

$$(a_1,a_2) \to (b_1,b_2) = (a_1 \to_G b_1, b_2 \prec_G a_2)$$
$$(a_1,a_2) \prec (b_1,b_2) = (a_1 \prec_G b_1, b_2 \to_G a_2)$$
$$\sim(a_1,a_2) = (\sim_G a_1, -_G a_2)$$



$\Gamma \vDash_{G^2_{(1,1)^\uparrow}(\to)} \alpha$ defined as preservation of $(1,1)^\uparrow$.
Its $(\wedge, \vee, \neg)$-fragment coincides with BD.

case (a): $G^2_{(1,0)}(\rightarrow)$, comparative uncertainty

$G^2_{(1,0)}(\rightarrow)$: **bi-Gödel logic expanded with a bi-lattice negation**

Axiomatization: bi-IL in the language $\{\wedge, \vee, \rightarrow, \prec, 0, 1\}$ extended with the prelinearity axiom: $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$

$$\neg\neg\alpha \leftrightarrow \alpha \quad \neg 0 \leftrightarrow {\sim} 0$$

$$\neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$$

$$\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$$

$$\neg(\alpha \rightarrow \beta) \leftrightarrow (\neg\beta \prec \neg\alpha)$$

$$\alpha \vdash {\sim}\neg\alpha$$

- $\neg$-negation normal form; $p \wedge \neg p \vdash q$
- Deduction theorem: $\Gamma, \alpha \vdash \beta$ iff $\Gamma \vdash {\sim}{-}\alpha \wedge {\sim}\neg\alpha \rightarrow \beta$
- Standard strong completeness (SSC)
- **Its theorems coincide with Wansing's I_4C_4 extended with prelinearity axiom.**

case (a): $G^2_{(1,1)^\uparrow}(\rightarrow)$, comparative uncertainty

Axiomatization: bi-IL in the language $\{\wedge, \vee, \rightarrow, \prec, 0, 1\}$ extended with the prelinearity axiom: $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$

$$\neg\neg\alpha \leftrightarrow \alpha \quad \neg 0 \leftrightarrow {\sim}0$$

$$\neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$$

$$\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$$

$$\neg(\alpha \rightarrow \beta) \leftrightarrow (\neg\beta \prec \neg\alpha)$$

$$\vdash \alpha \;/\; \vdash {\sim}\neg\alpha$$

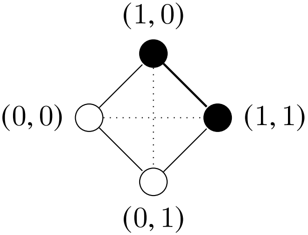
- $\neg$-negation normal form; $p \wedge \neg p \nvdash q$
- Deduction theorem: $\Gamma, \alpha \vdash \beta$ iff $\Gamma \vdash {\sim}{-}\alpha \rightarrow \beta$
- Standard strong completeness (SSC)
- **= Wansing's I_4C_4 extended with prelinearity axiom.**

Case (b): $G^2_{(1,1)^\uparrow}(\rightarrowtriangle)$, comparative uncertainty

twist product $[0,1]^{\bowtie}$, $\neg(a_1,a_2) = (a_2,a_1)$, $F = (1,1)^{\uparrow}$:

$[0,1]^{\bowtie}$ expanded with

$$(a_1,a_2) \rightarrowtriangle (b_1,b_2) = (a_1 \rightarrow_G b_1, a_1 \wedge b_2)$$
$$\sim(a_1,a_2) = (\sim_G a_1, a_1)$$



$\Gamma \vDash_{G^2_{(1,1)^\uparrow}(\rightarrowtriangle)} \alpha$ defined as preservation of $F = \{(1,a) \mid a \in [0,1]\}$.

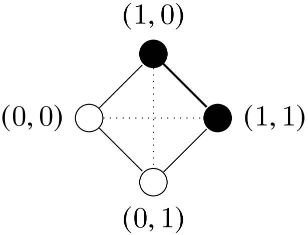
The weak equivalence $\alpha \leftarrowtriangle\!\!\rightarrowtriangle \beta := (\alpha \rightarrowtriangle \beta) \wedge (\beta \rightarrowtriangle \alpha)$ is not congruential, the strong one $\alpha \Longleftrightarrow \beta := (\alpha \leftarrowtriangle\!\!\rightarrowtriangle \beta) \wedge (\neg\alpha \leftarrowtriangle\!\!\rightarrowtriangle \neg\beta)$ is.

Case (b): $G^2_{(1,1)^\uparrow}(\rightarrowtriangle)$, comparative uncertainty

twist product $[0,1]^{\bowtie}$, $\neg(a_1,a_2) = (a_2,a_1)$, $F = (1,1)^\uparrow$:

$[0,1]^{\bowtie}$ expanded with

$$(a_1,a_2) \rightarrowtriangle (b_1,b_2) = (a_1 \rightarrow_G b_1, a_1 \wedge b_2)$$
$$\sim(a_1,a_2) = (\sim_G a_1, a_1)$$



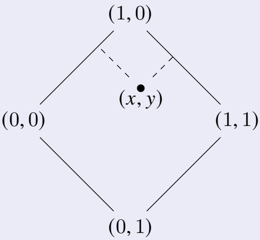
$\Gamma \vDash_{G^2_{(1,1)^\uparrow}(\rightarrowtriangle)} \alpha$ defined as preservation of $F = \{(1,a) \mid a \in [0,1]\}$.

The resulting logic coincides with Nelson's $N4^\perp$ extended with prelinearity (global consequence).

Two-dimensional logics: summing up

... of quantified uncertainty

- $Ł^2_{(1,1)^{\uparrow}}(\rightarrowtriangle) = \mathrm{N}Ł$, $Ł^2_{(1,0)^{\uparrow}}(\rightarrow) = Ł^2$, $Ł^2_{(1,1)^{\uparrow}}(\rightarrow)$
- FSSC, SC w.r.t. twist products of MV algebras (MV-chains)
- Varying the filters $(x, y)^{\uparrow}$: different tautologies, different entailments
- Constraint tableaux calculi, finitary entailment is coNP-complete.



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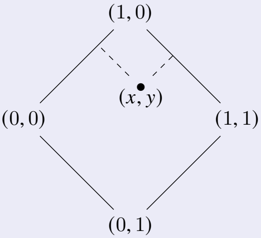
- M.B., S. Frittella, D. Kozhemiachenko, O. Majer, S. Nazari. *Reasoning with belief functions over Belnap–Dunn logic*. submitted.

Two-dimensional logics: summing up

... of comparative uncertainty

- $G^2_{(1,0)^\uparrow}(\rightarrow)$, $G^2_{(1,1)^\uparrow}(\rightarrow)$, $G^2_{(1,1)^\uparrow}(\rightarrowtriangle)$
- SSC, SC w.r.t. twist products of G-algebras (G-chains) or bi-G algebras (bi-G chains)
- Frame semantics
- Varying the filters $(x, y)^\uparrow$: same tautologies, different entailments:
- Constraint tableaux calculi, frame semantics, finitary entailment is coNP-complete.



$1 > x > y > 0$ for $G^2(\rightarrow)$

$\models_{(x,1)^\uparrow} \subset \models_{(1,1)^\uparrow} \subset \models_{(1,0)^\uparrow}$

$\models_{(x,1)^\uparrow} \subset \models_{(y,x)^\uparrow} \subset \models_{(x,x)^\uparrow} \subset \models_{(x,y)^\uparrow} \subset \models_{(1,0)^\uparrow}$

An application: two-layer logics of probabilities

Belnapian probabilities (quantified uncertainty)

- Two-layer logics $(\mathsf{BD}, M_p, Ł^2)$, $(\mathsf{BD}, M_p, Ł^2\triangle)$, or $(\mathsf{BD}, M_p^N, \mathsf{N}Ł)$
- Finite strong completeness w.r.t. intended semantics:

$$\text{Models: } \quad \langle S, \Vdash^+, \Vdash^-, \mathrm{p} : \mathcal{P}S \to [0,1] \rangle$$
$$\text{Semantics: } \quad |P\varphi| := (\mathrm{p}(|\varphi|^+), \mathrm{p}(|\varphi|^-))$$

Modal axioms

$$M_p: \quad \vdash_{Ł^2} P\neg\varphi \leftrightarrow \neg P\varphi \qquad \{\vdash_{Ł^2} P\varphi \to P\psi \mid \varphi \vdash_{\mathsf{BD}} \psi\}$$

$$\vdash_{Ł^2} P(\varphi \vee \psi) \leftrightarrow (P\varphi \ominus P(\varphi \wedge \psi)) \oplus P\psi$$

- M.B., S. Frittella, O. Majer, S. Nazari. *Belief based on inconsistent information*, DaLi 2020, LNCS volume 12569, pp 68-86, 2020.
- M.B., S. Frittella, D. Kozhemiachenko, O. Majer, S. Nazari. *Reasoning with belief functions over Belnap–Dunn logic*. submitted.

An application: two-layer logics of probabilities

Belnapian probabilities (quantified uncertainty)

- Two-layer logics (BD, M_p, $Ł^2$), (BD, M_p, $Ł^2\triangle$), or (BD, M_p^N, NŁ)
- Finite strong completeness w.r.t. intended semantics:

$$\text{Models: } \langle S, \Vdash^+, \Vdash^-, \mathrm{p} : \mathcal{P}S \to [0,1] \rangle$$
$$\text{Semantics: } |P\varphi| := (\mathrm{p}(|\varphi|^+), \mathrm{p}(|\varphi|^-))$$

Modal axioms

M_p^N:

$$\vdash_{\text{NŁ}} P\neg\varphi \leftrightarrow \neg P\varphi \quad \{\vdash_{\text{NŁ}} P\varphi \Rightarrow P\psi \mid \varphi \vdash_{\text{BD}} \psi\}$$
$$\vdash_{\text{NŁ}} P(\varphi \vee \psi) \leftrightarrow (P\varphi \ominus P(\varphi \wedge \psi)) \oplus P\psi$$
$$\vdash_{\text{NŁ}} P(\varphi \wedge \psi) \leftrightarrow (P\varphi \ominus P(\varphi \vee \psi)) \oplus P\psi$$

- M.B., S. Frittella, O. Majer, S. Nazari. *Belief based on inconsistent information*, DaLi 2020, LNCS volume 12569, pp 68-86, 2020.
- M.B., S. Frittella, D. Kozhemiachenko, O. Majer, S. Nazari. *Reasoning with belief functions over Belnap–Dunn logic*. submitted.

Example: measuring $(\varphi \wedge \neg\varphi)$

For a BD formula φ,

NŁ

- $P(\varphi \wedge \neg\varphi) \rightarrowtriangle {\sim}(P(\varphi \wedge \neg\varphi))$ says ”rather small degree of conflict” (closer to 0 then 1)
- ${\sim}P(\varphi \wedge \neg\varphi) \rightarrowtriangle (P(\varphi \wedge \neg\varphi))$ says ”rather big degree of conflict” (closer to 1 then 0)

$Ł^2$

- $P(\varphi \wedge \neg\varphi) \to {\sim}(P(\varphi \wedge \neg\varphi))$ says ”rather small degree of conflict” and ”rather small degree of ignorance”

(By ”says” we mean consequences of the formula being designated in the resp. algebra.)

Example: reasoning about $(\varphi \wedge \neg\varphi)$ in $Ł^2$

Assume $\sim P(\varphi \wedge \neg\varphi)$:

$$\sim P(\varphi \wedge \neg\varphi) \vdash \sim\neg\sim P(\varphi \wedge \neg\varphi) \vdash \sim\sim\neg P(\varphi \wedge \neg\varphi) \vdash \neg P(\varphi \wedge \neg\varphi) \vdash P(\varphi \vee \neg\varphi).$$

From (A3) we know that $\vdash (P\varphi \ominus P(\varphi \wedge \neg\varphi)) \oplus P\neg\varphi$
which is equivalent to

$$\vdash (P\varphi \to P(\varphi \wedge \neg\varphi)) \to P\neg\varphi.$$

From $\sim P(\varphi \wedge \neg\varphi) \vdash (P\varphi \to P(\varphi \wedge \neg\varphi)) \leftrightarrow \sim P\varphi$ we obtain

$$\sim P(\varphi \wedge \neg\varphi) \vdash \sim P\varphi \to \neg P\varphi \vdash \sim\neg P\varphi \to P\varphi.$$

As $\sim\neg P\varphi \to P\varphi$ and $P\varphi \to \sim\neg P\varphi$ are inter-derivable, we see that assuming $\sim P(\varphi \wedge \neg\varphi)$ entails that $P\varphi$ is classical.
On the other hand, assuming $P\varphi$ is classical, we can prove that $\sim P(\varphi \wedge \neg\varphi) \leftrightarrow P(\varphi \vee \neg\varphi)$.

Expressing belief function and plausibility axioms

We define a sequence of outer ($Ł^2$) formulas γ_n in propositional letters of the inner language $p_1, \ldots, p_n$ inductively as follows:

$$\gamma_1 := Bp_1$$
$$\gamma_{n+1} := \gamma_n \oplus (Bp_{n+1} \ominus \gamma_n[B\psi : B(\psi \wedge p_{n+1}) \mid B\psi \text{ atoms of } \gamma_n])$$

The n-monotonicity axiom

is expressed by substitution instances of

$$\alpha_n := \gamma_n \rightarrow B(\bigvee_{i=1}^{n} p_n).$$

Expressing belief function and plausibility axioms

We define sequences of outer (NŁ) formulas γ_n, σ_n in propositional letters of the inner language $p_1, \ldots, p_n$ inductively as follows:

$$\begin{aligned}
\gamma_1 &\coloneqq Bp_1 \\
\gamma_{n+1} &\coloneqq \gamma_n \oplus (Bp_{n+1} \ominus \gamma_n[B\psi : B(\psi \wedge p_{n+1}) \mid B\psi \text{ atoms of } \gamma_n]) \\
\sigma_1 &\coloneqq Plp_1 \\
\sigma_{n+1} &\coloneqq \sigma_n \oplus (Plp_{n+1} \ominus \sigma_n[Pl\psi : Pl(\psi \vee p_{n+1}) \mid Pl\psi \text{ atoms of } \sigma_n])
\end{aligned}$$

The n-th belief function and plausibility axioms

are expressed by substitution instances of

$$\alpha_n \coloneqq \gamma_n \rightarrow B(\bigvee_{i=1}^{n} p_n).$$

$$\beta_n \coloneqq Pl(\bigwedge_{i=1}^{n} p_n) \rightarrow \sigma_n.$$

An application: two-layer logics of belief functions and plausibilities

Belief (quantified uncertainty)

- Two-layer logic $(\mathsf{BD}, M_b, Ł^2)$
- Finite strong completeness w.r.t. intended semantics

Models: $\langle S, \Vdash^+, \Vdash^-, \mathtt{bel} : \mathcal{P}S \to [0,1] \rangle$

Semantics: $|B\varphi| := (\mathtt{bel}(|\varphi|^+), \mathtt{bel}(|\varphi|^-))$

Modal axioms

M_b: $\vdash_{Ł^2} B\neg\varphi \leftrightarrow \neg B\varphi$ $\quad \{\vdash_{Ł^2} B\varphi \to B\psi \mid \varphi \vdash_{\mathsf{BD}} \psi\}$

$\{\vdash_{Ł^2} \alpha_n \mid n \in N\}$

M.B., S. Frittella, D. Kozhemiachenko, O. Majer, S. Nazari. *Reasoning with belief functions over Belnap–Dunn logic*. submitted.

An application: two-layer logics of belief functions and plausibilities

Belief and plausibility

- Two-layer logic $(\mathrm{BD}, M_b^N, \mathrm{N}Ł)$
- Finite strong completeness w.r.t. intended semantics

$$\text{Models: } \langle S, \Vdash^+, \Vdash^-, \mathtt{bel}, \mathtt{pl} : \mathcal{P}S \to [0,1] \rangle$$

$$\text{Semantics: } |B\varphi| := (\mathtt{bel}(|\varphi|^+), \mathtt{pl}(|\varphi|^-))$$

$$|Pl\varphi| := (\mathtt{pl}(|\varphi|^+), \mathtt{bel}(|\varphi|^-))$$

Modal axioms

$$M_b^N: \quad \vdash_{\mathrm{N}Ł} Pl\neg\varphi \Leftrightarrow \neg B\varphi \quad \{\vdash_{\mathrm{N}Ł} B\varphi \Longrightarrow B\psi, Pl\varphi \Longrightarrow Pl\psi \mid \varphi \vdash_{\mathrm{BD}} \psi\}$$

$$\{\vdash_{\mathrm{N}Ł} \alpha_n, \vdash_{\mathrm{N}Ł} \beta_n \mid n \in N\}$$

M.B., S. Frittella, D. Kozhemiachenko, O. Majer, S. Nazari. *Reasoning with belief functions over Belnap–Dunn logic*. submitted.

Two-layer logics of comparative uncertainty

- Two-layer logic $(\mathsf{BD}, M_c, G^2(\to))$
- Strong completeness w.r.t. intended semantics

Models: $\langle S, \Vdash^+, \Vdash^-, \pi : \mathcal{P}S \to [0,1] \rangle$
Semantics: $|C\varphi| := (\pi(|\varphi|^+), \pi(|\varphi|^-))$

Modal axioms

M_c: $\vdash_{G^2} C\neg\varphi \leftrightarrow \neg C\varphi$ $\{\vdash_{G^2} C\varphi \to C\psi \mid \varphi \vdash_{\mathsf{BD}} \psi\}$

- similarly for $(\mathsf{BD}, M_c^N, G^2(\twoheadrightarrow))$
- bi-G and $G^2(\to)$ can be also used to capture two-layer logics of qualitative uncertainty measures (probabilities):

$$\varphi \lesssim \psi := \triangle(B\varphi \to B\psi)$$

M.B., S. Frittella, D. Kozhemiachenko, O. Majer, *Comparing certainty in contradictory evidence*. manuscript.

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