Initiating descent theory for closure spaces

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Topology, Algebra and Categories in Logic (TACL 2022),
June 20 - 24, 2022, University of Coimbra.

Closure spaces

By a *closure space* we will mean a pair $(A, \mathcal{C}_A)$, in which A is a set and $\mathcal{C}_A$ a set of subsets of A closed under arbitrary intersections.

We will consider the category **CLS** of closure spaces, where a morphism $\alpha : A \to B$ is a map α from A to B with

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A closure space structure $\mathcal{C}_A$ on a set A can be equivalently described as a closure operator on the power set $\mathrm{P}(A)$ of A written as $X \mapsto \overline{X}$ (or, more precisely, as $X \mapsto \overline{X}^A$) and satisfying

$$X \subseteq X' \Rightarrow \overline{X} \subseteq \overline{X'}, \;\; X \subseteq \overline{X}, \;\; \overline{\overline{X}} = \overline{X}.$$

The relationship between these two types of structures is given by

$$\overline{X} = \bigcap_{X \subseteq A' \in \mathcal{C}} A' \;\text{ and }\; X \in \mathcal{C} \Leftrightarrow X = \overline{X}.$$

CLS is a topological category

The underlying set functor $U : \mathbf{CLS} \to \mathbf{Sets}$ is *topological* and so we know, in particular, how to construct pullbacks and coequalizers:

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Lemma

A pullback in **CLS** *is a diagram of the form*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B \end{array}$$

where $E \times_B A$ *is the set* $\{(e, a) \in E \times A \mid p(e) = \alpha(a)\}$, *and*

$$\mathcal{C}_{E \times_B A} = \{E' \times_B A' = \pi_1^{-1}(E') \cap \pi_2^{-1}(A') \mid E' \in \mathcal{C}_E \,\&\, A' \in \mathcal{C}_A\}.$$

Lemma

A morphism $p : E \to B$ *in* **CLS** *is a regular epimorphism if and only if* p *is a surjective map with* $\mathcal{C}_B = \{B' \subseteq B \mid p^{-1}(B') \in \mathcal{C}_E\}$. □

Some classes of morphisms in CLS

Proposition

For closure spaces E and B, and a map $p : E \to B$, the following conditions are equivalent:

(a) $p : E \to B$ *is a morphism in* **CLS**;

(b) $\overline{p^{-1}(X)} \subseteq p^{-1}(\overline{X})$ *for every* $X \subseteq B$;

(c) $p(\overline{p^{-1}(X)}) \subseteq \overline{X}$ *for every* $X \subseteq B$;

(d) $p(\overline{Y}) \subseteq \overline{p(Y)}$ *for every* $Y \subseteq E$;

(e) $\overline{Y} \subseteq p^{-1}(\overline{p(Y)})$ *for every* $Y \subseteq E$.

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The following conditions on a morphism $p : E \to B$ in **CLS** *are equivalent:*

(a) p *is closed, that is,* $Y \in \mathcal{C}_E \Rightarrow p(Y) \in \mathcal{C}_E$;

(b) $p(\overline{Y}) \supseteq \overline{p(Y)}$ *for every* $Y \subseteq E$;

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(a) p *is open, that is,* $-Y \in \mathcal{C}_E \Rightarrow -p(Y) \in \mathcal{C}_B$*;*

(b) $\overline{X} \subseteq -p(-\overline{p^{-1}(X)})$ *for every* $X \subseteq B$*;*

(c) $\overline{p^{-1}(X)} \supseteq p^{-1}(\overline{X})$ *for every* $X \subseteq B$*;*

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Pullback and pullback stable reg. epim. in CLS

Proposition

Consider again the pullback diagram for (p, α)*. For* $Z \subseteq E \times_B A$ *one has* $\overline{Z} = \pi_1^{-1}(\overline{\pi_1(Z)}) \cap \pi_2^{-1}(\overline{\pi_2(Z)})$.

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Descent with respect to the basic fibration

For a morphism $p\colon E \to B$ in a category $\mathbb{C}$ with pullbacks and coequalizers of equivalence relations let $T^p = (T^p, \eta^p, \mu^p)$ be the monad induced in $\mathbb{C} \downarrow E$ by the adjunction

$$p! \dashv p^*\colon \mathbb{C} \downarrow B \to \mathbb{C} \downarrow E.$$

Then descent data for p are the T^p-algebras and the category of descent data $Des(p)$ is the Eilenberg-Moore category of algebras $(\mathbb{C} \downarrow E)^{T^p}$.

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Let $K^p\colon \mathbb{C} \downarrow B \to (\mathbb{C} \downarrow E)^T = Des(p)$ be the comparison functor.

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Definition

The morphism $p\colon E \to B$ is said to be
- a descent morphism if K^p is fully faithful;
- an effective descent morphism if K^p is a category equivalence, that is, if p^* is monadic.

General results on descent

The left adjoint L^p of K^p is defined by $L^p(C,\gamma,\xi) = (Q,\delta)$ where $q = coeq(\xi,\pi_2)$ and δ is the induced morphism

$$\begin{array}{ccccc} E \times_B C & \overset{\xi}{\underset{\pi_2}{\rightrightarrows}} & C & \xrightarrow{q} & Q \\ {\scriptstyle E\times_B\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ E \times_B E & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & E & \xrightarrow[p]{} & B \end{array}$$

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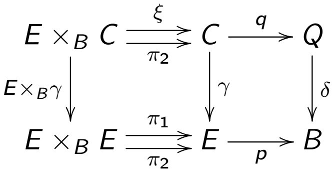

Proposition

A morphism in $\mathbb{C}$ is a descent morphism if and only if it is a pullback stable regular epimorphism.

A descent morphism in $\mathbb{C}$ is an effective descent morphism if and only if for every descent morphism $p\colon E \to B$ and every diagram as above, γ is an isomorphism when δ is an isomorphism.

Descent versus effective descent

In **CLS** there exist descent morphisms that are not effective descent if and only if the following situation may occur: for a descent morphism p there exist a commutative diagram

$$\begin{array}{ccccc} E\times_B E' & \overset{\xi}{\underset{\pi_2}{\rightrightarrows}} & E' & \xrightarrow{q} & B \\ {\scriptstyle E\times_B\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle 1_B} \\ E\times_B E & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & E & \xrightarrow[p]{} & B \end{array} ,$$

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(constructed as above) where $U(\gamma)$ is the identity map and $E' \neq E$.

That is, if

1. the identity map $\gamma \colon E' \to E$ belongs to **CLS**,
2. there exists descent data (E', γ, ξ) for p, and
3. $U(q) = U(p)$.

Existence of such descent data for p

Lemma

Suppose the identity map $1_E : E' \to E$ is a morphism in **CLS**. *Then the following conditions are equivalent:*

(a) *there exists a descent data for p of the form $(E', 1_E, \xi)$;*

(b) *there exists a unique descent data for p of the form $(E', 1_E, \xi)$;*

(c) *the triple $(E', 1_E, \pi_1)$, where $\pi_1 : E \times_B E' \to E'$ is defined by $\pi_1(e, e') = e$, is a descent data for p;*

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(e) $\overline{Y} \cap p^{-1}(p(\overline{p^{-1}(p(Y))}')) \subseteq \overline{Y}'$ *for all* $Y \subseteq E$;

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(g) $\overline{Y} \cap p^{-1}(p(\overline{p^{-1}(p(Y))}')) \subseteq Y$ *for all* $Y \in \mathcal{C}_{E'}$;

(h) $\overline{Y} \cap p^{-1}(p(\overline{p^{-1}(p(Y))}')) = Y$ *for all* $Y \in \mathcal{C}_{E'}$.

Descent = effective descent for finite closure spaces

Lemma

Suppose the equivalent conditions of the previous Lemma are satisfied and let us write p' *for* p *considered as a morphism from* E' *to* B. *If both* p *and* p' *are regular epimorphisms, then, for every* $Y \in \mathcal{C}_{E'} \setminus \mathcal{C}_E$, *there exists* $Y^* \in \mathcal{C}_{E'} \setminus \mathcal{C}_E$ *with* $Y \subset Y^*$. *In particular, if* $\mathcal{C}_{E'} \neq \mathcal{C}_E$, *then* E *is infinite.*

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Theorem

Every descent morphism in the category **FCLS** *of finite closure spaces is an effective descent morphism.*

Preorders as closure spaces

$$\textbf{Preord} \to \textbf{Top} \to \textbf{CLS},$$

are full inclusions where **Preord** is the category of preordered sets and **Top** is the category of topological spaces.
Considering a preorder B as either a topological space or a closure space, for any $X \subseteq B$, we have

$$\overline{X} = \uparrow X = \{b \in B \mid x \leqslant b \text{ for some } x \in X\}.$$

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Not every descent morphism in **Preord** is a descent morphism in **Top**. (M.M. Clementino and G. Janelidze, 2020)

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A morphism in **Preord** *is a descent morphism in* **Preord** *if and only if it is a descent morphism in* **CLS**.

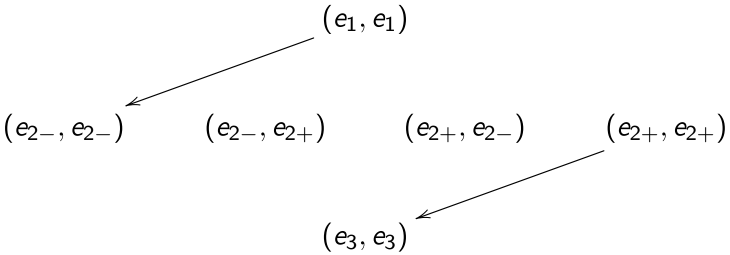
Descent does not coincide with effective descent in **Preord**, not even in **FPreord** where we have $\mathbf{FPreord} \sim \mathbf{FTop} \to \mathbf{FCLS}$.

A non-effective descent morphism

Let $p : E \to B$ be the morphism in **FPreord**, where

- $B = \{b_1, b_2, b_3\}$ is the ordered set with $b_1 < b_2 < b_3$.
- $E = \{e_1, e_{2-}, e_{2+}, e_3\}$ is the ordered set with $e_1 < e_{2-}$, $e_{2+} < e_3$, $e_1 < e_3$.
- $p(e_1) = b_1$, $p(e_{2-}) = b_2 = p(e_{2+})$, and $p(e_3) = b_3$.

For $E' = \{e_1, e_{2-}, e_{2+}, e_3\}$ with $e_1 < e_{2-}$, $e_{2+} < e_3$, the pullback $E \times_B E'$ can be presented as the diagram



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Then p is a non-effective descent morphism.

And for the corresponding closure spaces?

The first projection $\pi_1 : E \times_B E' \to E'$ is not a morphism in **CLS**.

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The set $Y = \{e_1, e_{2-}\}$ is closed in E' and

$$Z = \pi_1^{-1}(Y) = \{(e_1, e_1), (e_{2-}, e_{2-}), (e_{2-}, e_{2+})\},$$

is obviously closed in the pullback $E \times_B E'$ in **Preorder** but not in the pullback $E \times_B E'$ in **FCLS**: there $\overline{Z} \neq Z$:

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$$= \{(e_1, e_1), (e_{2-}, e_{2-}), (e_{2-}, e_{2+}), (e_3, e_3)\} \neq Z.$$

Furthermore, in the pullback $E \times_B E'$ in **FCLS**, putting $Z = U \cup V$ with $U = \{(e_1, e_1), (e_{2-}, e_{2-})\}$ and $V = \{(e_{2-}, e_{2+})\}$,

$$\overline{U} = U \text{ and } \overline{V} = V, \text{ while } \overline{U \cup V} \neq \overline{U} \cup \overline{V},$$

which is what could not happen in a preorder (since it could not happen in a topological space in general).

Connections with “strict monadic topology"

What we call “strict monadic topology” generalizes the category of compact Hausdorff spaces by replacing it with the category $\mathrm{Alg}(T)$ of algebras over an arbitrary monad T over the category of sets, and developing counterparts of topological notions in $\mathrm{Alg}(T)$.

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However, it turns out that it preserves and reflects descent and effective descent morphisms: since $\mathrm{Alg}(T)$ is Barr exact, to prove that is just to prove that every surjective closed map of closure spaces is an effective descent morphism in $\mathbf{CLS}$.

Surjective closed maps are effective descent morphisms

The result easily follows from a simple closure-space-variation of an old result of W. Tholen and M.S. (1991):

Theorem

A regular epimorphism p in a category $\mathbb{C}$ with pullbacks and coequalizers of equivalence relations is an effective descent morphism if and only if, for every T^p-algebra (C, γ, ξ) the equivalence relation (π_2, ξ) is effective and its coequalizer is a pullback stable regular epimorphism

We also proved there that the equivalence relation (π_2, ξ) is effective if and only if the left adjoint L^p of the comparison functor K^p is faithful, which is always the case in topological categories like **CLS**.

Surjective closed maps are effective descent morphisms

So, a regular epimorphism p in **CLS** is an effective descent morphism if and only if, for every T^p-algebra (C, γ, ξ) the coequalizer of (π_2, ξ) is a pullback stable regular epimorphism.

We have that

Lemma

The class of closed maps is pullback stable.

and from that we conclude the desired result:

Theorem

The surjective closed maps are effective descent morphisms.

We remark that the same holds for (surjective) open maps.

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