Projectivity in (bounded) commutative integral residuated lattices

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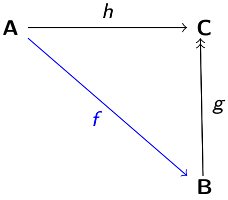
Projectivity (algebraic)

Given a class K of algebras, an algebra $\mathbf{A} \in \mathsf{K}$ is **projective** in K if for all $\mathbf{B}, \mathbf{C} \in \mathsf{K}$



An algebra $\mathbf{B}$ is a **retract** of an algebra $\mathbf{A}$ if there is an epimorphism $g : \mathbf{A} \longmapsto \mathbf{B}$ and a homomorphism $f : \mathbf{B} \longmapsto \mathbf{A}$ with $gf = \mathrm{id}_{\mathbf{B}}$ (and thus f is necessarily injective).

In (quasi)varieties projective algebras = retracts of free algebras (Whitman).

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A finitely presented algebra $\mathbf{A}$ in a variety V is a quotient of a finitely generated free algebra $\mathbf{F}_{\mathsf{V}}(n)$ by a compact congruence θ, i.e. $\mathbf{A} \cong \mathbf{F}_{\mathsf{V}}(n)/\theta$.

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Being finitely presented and being finitely generated are preserved by categorical equivalences in algebraic categories (Gabriel, Ulmer).

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- (folklore) an abelian group is projective in the variety of abelian groups if and only if it is free;
- (Beynon) a finitely generated abelian ℓ-group is projective in the variety of abelian ℓ-groups if and only if it is finitely presented.

The mandatory slide

A **commutative and integral residuated lattice** (a CIRL) is an algebra $\langle A, \vee, \wedge, \cdot, \rightarrow, 1\rangle$ such that

1. $\langle A, \vee, \wedge, 1\rangle$ is a lattice with a top element 1;
2. $\langle A, \cdot, 1\rangle$ is a commutative monoid;
3. $(\cdot, \rightarrow)$ form a residuated pair w.r.t. the lattice ordering, i.e. for all $a, b, c \in A$

$$ab \leq c \qquad \text{if and only if} \qquad a \leq b \rightarrow c.$$

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$\mathsf{FL}_{\mathsf{ew}}$-algebras are bounded CIRLs: they have an extra constant 0 that is the least element of the lattice.
The variety of $\mathsf{FL}_{\mathsf{ew}}$-algebras is the equivalent algebraic semantics of the Full Lambek calculus with the structural rules of exchange and weakening.

FL_{ew}-algebras

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In fact:

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(A.-Ugolini) The only finite projective FL_{ew} *algebra is* $\mathbf{2}$.

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The only splitting algebra in FL_{ew} is 2 (Kowalski-Ono).

But, as the free algebra over the empty set, 2 is projective in every variety of FL_{ew}-algebras, and the thesis follows. □

Hoops

There are two equations in the language of CIRLs that bear interesting consequences, i.e.

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If a variety satisfies the divisibility condition (div) then the lattice ordering becomes the inverse divisibility ordering: for any algebra $\mathbf{A}$ therein and for all $a, b \in A$

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- in fact every variety of basic hoops is a variety of CIRLs;
- a well investigated example of a variety of hoops that is not a variety of CIRLs is the variety of **Brouwerian semilattices**.

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Since (Blok-Ferreirim) the locally finite variety of hoops are exactly the varieties in which the monoidal operation is n-potent for a fixed n (i.e. they satisfy $x^n \approx x^{n+1}$) it follows at once that the finite Brouwerian semilattices are exactly the finitely presented projective ones (as observed by Ghilardi using a different argument).

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Moreover every locally finite prelinear variety of CIRLs must have the same property, since it is (term equivalent to) a locally finite variety of basic hoops.

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(A.-Ugolini) The finitely presented cancellative hoops are exactly the finitely generated and projective ones.

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Since there are finite bounded hoops (e.g. the three element MV-algebra) of which 2 is not a homomorphic image, the result cannot hold if a bound is present.

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Since every finite algebra **A** in a variety is a subdirect product of finite subdirectly irreducible algebras, and all the subdirect factors are homomorphic images of **A**, we can sharpen a little our results.

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Let V *be a locally finite variety of bounded hoops or BL-algebras such that every finite subdirectly irreducible in* V *has* $\mathbf{2}$ *as homomorphic image. Then every finitely presented algebra in* V *is projective.*

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This is (yet another) reason why every finitely presented (i.e., finite) Boolean algebra is projective: the variety of Boolean algebras is locally finite and the only subdirectly irreducible is $\mathbf{2}$.

A more intriguing example is the following: a variety of FL_{ew}-algebras is **Stonean** if it satisfies the equation $\neg x \vee \neg\neg x \approx 1$ (of course $\neg x := x \to 0$).

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It is the straightforward consequence of the characterization of the subdirectly irreducible BL-algebras in (A.-Montagna) that a finite subdirectly irreducible algebra in a Stonean variety is of the form $A = 2 \oplus \mathbf{B}$, where $\mathbf{B}$ is a totally ordered hoop.

Since $\mathbf{B}$ is a filter of $\mathbf{A}$, we can collapse it and get 2 as a homomorphic image of $\mathbf{A}$. Hence Stonean BL-algebras fall under the scope of the previous result.

Stonean BL-algebras are a particular instance of a construction known as **generalized rotation**; projectivity in varieties of BL-algebras that are generalized rotations of varieties of basic hoops has been investigated by Sara and me in a separate paper.

Algebraic unification Ghilardi-style

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(Ghilardi's solution) A **unification problem** for a variety V is a finitely presented algebra $\mathbf{A} \in \mathsf{V}$; a **solution** is a homomorphism $u : \mathbf{A} \longrightarrow \mathbf{P}$, where $\mathbf{P}$ is a projective algebra in V. In this case u is called a **unifier** for $\mathbf{A}$ and we say that $\mathbf{A}$ is **unifiable**.

Unification types

If u_1, u_2 are unifiers for an algebra $\mathbf{A}$ (with projective targets $\mathbf{P}_1$ and $\mathbf{P}_2$) we say that u_1 is more general than u_2 if there exists a homomorphism $m : \mathbf{P}_1 \longrightarrow \mathbf{P}_2$ such that $mu_1 = u_2$.

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If all the $U_\mathbf{A}$ have a unique maximal element then the type of V is **unitary**; if in any case this maximal element is the identity, then V has **strong unitary** type.

Lemma

Let V *be any variety; then the following are equivalent.*

1. V *has strong unitary type;*
2. *for any finitely presented algebra* $\mathbf{A} \in \mathsf{V}$*,* $\mathbf{A}$ *is unifiable if and only if it is projective.*

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In varieties of FL_{ew}-algebras any unifiable algebra must have a surjective homomorphism on the two element algebra 2 and since, 2 is projective in every variety of FL_{ew}-algebras, we get:

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Let V *be any variety; then the following are equivalent.*

1. V *has strong unitary type;*
2. *for any finitely presented algebra* $\mathbf{A} \in \mathsf{V}$*,* $\mathbf{A}$ *is unifiable if and only if it is projective.*

In varieties of FL_{ew}-algebras any unifiable algebra must have a surjective homomorphism on the two element algebra $\mathbf{2}$ and since, $\mathbf{2}$ is projective in every variety of FL_{ew}-algebras, we get:

Lemma

For a variety V *of* FL_{ew}*-algebras the following are equivalent:*

1. V *has strong unitary type;*
2. *for any finitely presented* $\mathbf{A} \in \mathsf{V}$*,* $\mathbf{A}$ *has* $\mathbf{2}$ *as a homomorphic image if and only if* $\mathbf{A}$ *is projective.*

Theorem

The following varieties, and their corresponding logics, have strong unitary unification type:

1. *all locally finite subvarieties of hoops;*
2. *all locally finite subvarieties of bounded hoops and BL-algebras;*
3. *cancellative hoops.*

THANK YOU!