A representation theorem for a system of point-free geometry

(on-going work)

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TACL 2022

Outline

1. Introduction
2. Oval structures
3. Basic geometrical notions
4. The representation theorem

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1 Introduction

2 Oval structures

3 Basic geometrical notions

4 The representation theorem

Introduction

- What do I mean by geometry?
- What do I mean by point-free geometry?
- What is the objective of this talk?
- What the mood of the presentation will be?

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1 Introduction

2 Oval structures

3 Basic geometrical notions

4 The representation theorem

Basic notions

Let me focus on structures $\langle \mathbf{R}, \leq, \mathbf{O} \rangle$ such that:

- elements of $\mathbf{R}$ are called regions,
- $\leq \subseteq \mathbf{R}^2$ is part of relation,
- $\mathbf{O} \subseteq \mathbf{R}$ and its elements are called ovals (point-free analogs of certain convex sets).

First axioms

$$\langle \mathbf{R}, \leq \rangle \text{ is a complete atomless Boolean lattice.} \tag{O0}$$

$$\mathbf{O} \text{ is an algebraic closure system in } \langle \mathbf{R}, \leq \rangle \text{ containing } \mathbf{0}. \tag{O1}$$

$$\mathbf{O}^{+} \text{ is dense in } \mathbf{R}^{+}. \tag{O2}$$

Lines in the oval setting

Definition

By a line we understand a two element set $L = \{a, b\}$ of disjoint ovals, such that for any set of disjoint ovals $\{c, d\}$ with $a \leqslant c$ and $b \leqslant d$ it is the case that $a = c$ and $b = d$:

$$X \in \mathfrak{L} \overset{\text{df}}{\longleftrightarrow} \exists_{a,b \in \mathbf{O}^+} \Big(a \perp b \wedge X = \{a, b\} \wedge \forall_{c,d \in \mathbf{O}^+}(c \perp d \wedge a \leqslant c \wedge b \leqslant d \longrightarrow a = c \wedge b = d)\Big). \tag{df $\mathfrak{L}$}$$

For a line $L = \{a, b\}$ the elements of L will be called the sides of L.

Lines in the oval setting

Definition

Two lines $L_1 = \{a, b\}$ and $L_2 = \{c, d\}$ are paralell iff there is a side of L_1 which is disjoint from a side of L_2:

$$L_1 \parallel L_2 \stackrel{\text{df}}{\longleftrightarrow} \exists_{a \in L_1} \exists_{b \in L_2}\, a \perp b\,. \tag{df $\parallel$}$$

In case L_1 is not parallel to L_2 we say that L_1 and L_2 intersect and write '$L_1 \nparallel L_2$'.

Half-planes in the oval setting

Definition

A region x is a half-plane iff $x, -x \in \mathbf{O}^+$:

$$x \in \mathbf{H} \overset{\text{df}}{\longleftrightarrow} \{x, -x\} \subseteq \mathbf{O}^+ . \qquad (\mathtt{df}\,\mathbf{H})$$

Lines and half-planes in the oval setting

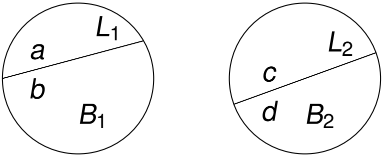

Figure: The structure $\mathbb{B}_2$.

Fact

B_1 and B_2 are the only half-planes of $\mathbb{B}_2$ and thus $\{B_1, B_2\}$ is the only line of $\mathbb{B}_2$ whose sides are half-planes. This line is parallel to every other line. In general, in $\mathbb{B}_n$ for $n \geqslant 2$ any pair $\{B_i, B_j\}$ with $i \neq j$ is a line parallel to every line in $\mathbb{B}_n$.

Specific axioms

Definition

A finite partition of the universe $\mathbf{1}$ is a set $\{x_1, \ldots, x_n\} \subseteq \mathbf{R}^+$ whose elements are pairwise disjoint and such that $\bigvee\{x_1, \ldots, x_n\} = \mathbf{1}$. For a partition $P = \{x_1, \ldots, x_n\}$ and $x \in \mathbf{R}^+$ by the partition of x induced by P we understand the following set:

$$\{x \cdot x_i \mid 1 \leqslant i \leqslant n \wedge x \cdot x_i \neq \mathbf{0}\}.$$

The sides of a line form a partition of $\mathbf{1}$; equivalently: the sides of a line are half-planes. (03)

Specific axioms

For any $a, b, c \in \mathbf{O}$ that are not aligned there is a line which separates a from $\mathrm{hull}(b + c)$. (O4)

Definition

$\mathrm{hull}\colon \mathbf{R} \longrightarrow \mathbf{R}$ is the operation given by:

$$\mathrm{hull}(x) := \bigwedge \{a \in \mathbf{O} \mid x \leq a\}. \qquad (\mathrm{df\,hull})$$

For $x \in \mathbf{R}$ the object $\mathrm{hull}(x)$ will be called the oval generated by x.

Specific axioms

If distinct lines L_1 and L_2 both cross an oval a, then they split a in at least three parts. (05)

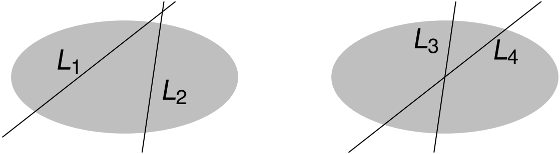

Figure: L_1 and L_2 split the oval into 3 parts, while L_3 and L_4 split it into 4 parts.

Specific axioms

No half-plane is part of any stripe and any angle. (06)

Thanks to (06) we can prove, e.g., that parallelity of lines is a Euclidean relation.

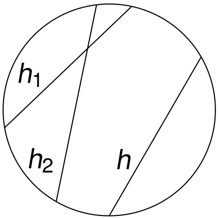

Figure: In Beltramy-Klein model: h is a part of the angle $h_2 \cdot -h_1$.

O-structures

Definition

A triple $\langle \mathbf{R}, \leq, \mathbf{O} \rangle$ is an O-structure iff $\langle \mathbf{R}, \leq, \mathbf{O} \rangle$ satisfies axioms (O∅)–(O6).

First theorem

Theorem

Let $\mathfrak{O} = \langle \mathbf{R}, \leqslant, \mathbf{O} \rangle$ be an O-structure and $\mathfrak{O}' := \langle \mathbf{R}, \leqslant, \mathbf{O}, \mathbf{H} \rangle$ be the structure obtained from $\mathfrak{O}$ by defining $\mathbf{H}$ as the set of all ovals whose complements are ovals. Then $\mathfrak{O}'$ satisfies all axioms for Śniatycki's geometry.

Now, towards the representation!

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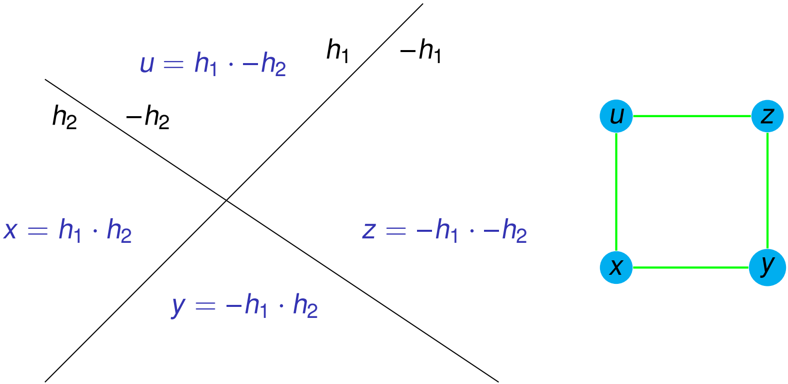
Pseudopoints

Definition

A pseudopoint is any net $(L_1 L_2)$ that contains four non-zero regions.

For any pseudopoint $(L_1 L_2)$, the lines L_1 and L_2 will be called its determinants. In case we have two pseudopoints $(L_1 L_2)$ and $(L_1 L_3)$ we say that they share a determinant L_1.

Pseudopoints



Points

Definition

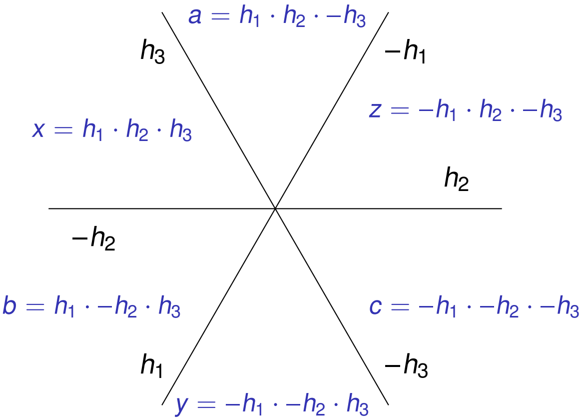
If $L_1, \ldots, L_k \in \mathfrak{L}$, an arbitrary element of the Cartesian product $L_1 \times \ldots \times L_k$ will be called an h-sequence. An h-sequence $\langle h_1, \ldots, h_k \rangle$ is non-zero iff $h_1 \cdot \ldots \cdot h_k \neq \mathbf{0}$, otherwise it is zero.

Definition

Lines L_1, L_2 and L_3 are tied iff $L_1 \times L_2 \times L_3$ contains two different zero and opposite h-sequences.

Points

$$\mathbf{0} = h_1 \cdot -h_2 \cdot -h_3 = -h_1 \cdot h_2 \cdot h_3$$



Points

Definition

A pseudopoint $(L_1 L_2)$ lies on L_3 iff L_1, L_2 and L_3 are tied.

Points

Definition

Psedopoints $(L_1 L_2)$ and $(L_3 L_4)$ are collocated (in symbols: $(L_1 L_2) \sim (L_3 L_4)$) iff $(L_1 L_2)$ lies on both L_3 and L_4.

Definition

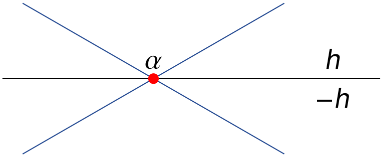
Collocation of pseudopoints is an equivalence relation, therefore points can be defined as its equivalence classes:

$$\Pi := \pi/_{\sim} . \qquad (\mathrm{df}\,\Pi)$$

Incidence relation

Definition

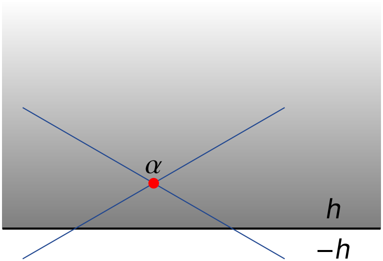
$\alpha \in \Pi$ is incident with a line L iff there is a pseudopoint $(L_1 L_2) \in \alpha$ such that $(L_1 L_2)$ lies on L.



Betweenneess relation

Definition

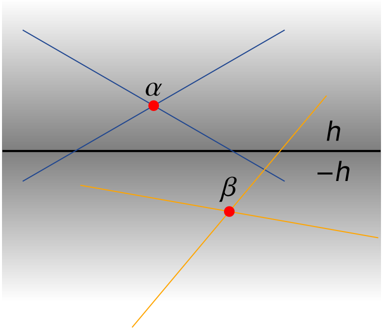
$\alpha \in \Pi$ lies in the half-plane h iff there is $(L_1 L_2) \in \alpha$ such that for every $x \in (L_1 L_2)$, $x \cdot h \neq \mathbf{0}$.



Betweenness relation

Definition

A line $L = \{h, -h\}$ lies between points α and β iff α lies in h and β lies in $-h$.



Betweenness relation

Definition

Points α, β and γ are collinear iff some three pseudpoints from, respectively, α, β and γ share a determinant L.

Definition

A point γ is between points α and β iff:

- α, β and γ are collinear and
- γ is incident with a line L which lies between α and β.

Second theorem

Theorem

Let $\langle \mathbf{R}, \leq, \mathbf{O} \rangle$ be an oval structure. Then individual notions of point and line and relational notions of incidence and betweenness are definable in such a way that the corresponding structure $\langle \Pi, \mathfrak{L}, \epsilon, \mathbf{B} \rangle$ satisfies all axioms of a second-order system of geometry of betweenness and incidence.

Crucial fact: Aleksander Śniatycki's theorem.

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The objective

- To find a representation of oval structures which in particular means
- to show that ovals are convex sets in a certain (point-based) space.

Toolbox

At the disposal we have:

- the standard basic incidence axioms,
- the standard betweenness axioms, including: Pasch axiom, Playfair axiom and the second-order continuity axiom.

Internal points of regions

Definition

- α lies in an oval a (or is an internal point of a) iff there is a pseudopoint $(L_1 L_2) \in \alpha$ such that for every $c \in (L_1 L_2)$, $c \cdot a \neq \mathbf{0}$.
- $\mathbf{Irl}(a)$ is the set of all internal points of a given oval a.
- I will write $(L_1 L_2) \lessdot x$ and $\alpha \lessdot x$ meaning, respectively, that the pseudopoint $(L_1 L_2)$ (the point α) is an internal pseudopoint (point) of x.

Internal point of regions

Fact

If $(L_1 L_2) \lessdot x$ *and* $(L_1 L_2) \sim (L_3 L_4)$, *then* $(L_3 L_4) \lessdot x$.

Theorem

There are no pseudopoint $(L_1 L_2)$ *and no half-plane* h *such that* $(L_1 L_2) \lessdot h$ *and* $(L_1 L_2) \lessdot -h$.

Idea of the proof.

Every net $(L_1 L_2 L_3)$ where the lines are pairwise distinct must contain the zero region. In case there is a half-plane h such that $(L_1 L_2)$ is an internal point of both h and its complement, then for $L = \{h, -h\}$, the net $(L_1 L_2 L)$ has eight non-zero regions, a contradiction. □

Topology

$$\forall_{a \in \mathbf{O}^+} \mathbf{Irl}(a) \neq \emptyset \quad (1)$$

$$\forall_{a,b \in \mathbf{O}} \mathbf{Irl}(a \cdot b) = \mathbf{Irl}(a) \cap \mathbf{Irl}(b)\,. \quad (2)$$

Fact

The set $\mathscr{B} := \left\{\mathbf{Irl}(a) \,\middle|\, a \in \mathbf{O}^+\right\}$ *is a basis.*

Definition

Let $\langle \Pi, \mathscr{O} \rangle$ be a topological space introduced via $\mathscr{B}$.

Properties of the topology

Theorem

The space Π is a Urysohn space.

Theorem

For every region x, $\mathbf{Irl}(x)$ is a regular open subset of Π, so Π is semi-regular.

Theorem

$\mathbf{Irl}\colon \mathbf{R} \to \mathbf{RO}(\Pi)$ *is a bijection.*

Ovals and convex sets

Definition

$\Gamma \subseteq \Pi$ is convex iff for every $\alpha, \beta \in \Gamma$, if γ is between α and β, then $\gamma \in \Gamma$.

The idea is to prove that for every oval *a*, the set of its internal points is convex in Π.

Ovals and convex sets

Lemma

Every oval is the infimum of the set of all half-planes of whose part it is.

Lemma

For every half-plane h, $\mathbf{Irl}(h)$ is convex in Π.

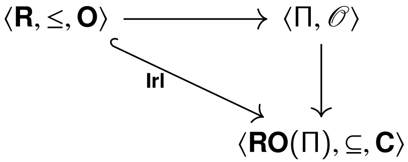
Theorem

For every $a \in \mathbf{O}$, $\mathbf{Irl}(a)$ is convex in Π.

Corollary

For every half-plane h, $\mathbf{Irl}(h)$ is a half-plane in Π, that is the Boolean complement of $\mathbf{Irl}(h)$ in $\mathbf{RO}(\Pi)$ is convex.

The representation theorem



Irl is a dense embedding.

An open problem

Theorem (Hypothesis)

For every convex open set $C \subseteq \Pi$ there is an oval a such that $\mathbf{Irl}(a) = C$.

Bibliography

Giangiacomo Gerla and Rafał Gruszczyński, **Point-free geometry, ovals and half-planes**, *Review of Symbolic Logic*, Volume 10, Issue 2 (2017), pp. 237–258

Rafał Gruszczyński and Andrzej Pietruszczak, **Fulldevelopment of Tarski's geometry of solids**, *Bulletin of Symbolic Logic*, Volume 14, No. 4 (2008), pp. 481–540

Victor Pambuccian, **Groups and plane geometry**, *Studia Logica*, 81:387–398 (2005)

Aleksander Śniatycki, **An axiomatics of non-Desarguean geometry based on the half-plane as the primitive notion**, *Dissertationes Mathematicae*, no. LIX, PWN, Warszawa, 1968

Alfred Tarski, **Foundations of the geometry of solids**, Logic, semantic, metamathematics, papers from 1923 to 1939, Clarendon Press, Oxford (1956), pp. 24–29

Support

Research supported by National Science Center, Poland, grant Logico-philosophical foundations of geometry and topology, no. 2020/39/B/HS1/00216.

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