Baker-Beynon duality beyond finite presentations

Serafina Lapenta
joint work with Luca Carai and Luca Spada

University of Salerno

TACL 2021-22

What we will talk about

- Preliminaries on the structures involved;
- Baker-Beynon duality and a general approach to 'affine dualities'
- Our results

Abelian ℓ-groups and vector lattices

A general approach

Beyond Baker-Beynon duality

ℓ-groups and vector lattices

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ℓ-groups and vector lattices form varieties.

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- A proper ℓ-ideal is called maximal if it is maximal wrt inclusion.
- A nontrivial ℓ-group/vector lattice A is simple if $\{0\}$ and A are the only ℓ-ideals of A.

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- A/I is simple iff I is maximal.
- A/I is semisimple iff I is intersection of maximal ℓ-ideals.

Piecewise linear functions

A continuous function $f : \mathbb{R}^\kappa \to \mathbb{R}$ is piecewise linear if there exist $g_1, \ldots, g_n$ linear homogeneous polynomials in the variables $(x_\alpha)_{\alpha<\kappa}$ such that for each $x \in \mathbb{R}^\kappa$ we have $f(x) = g_i(x)$ for some $i = 1, \ldots, n$.

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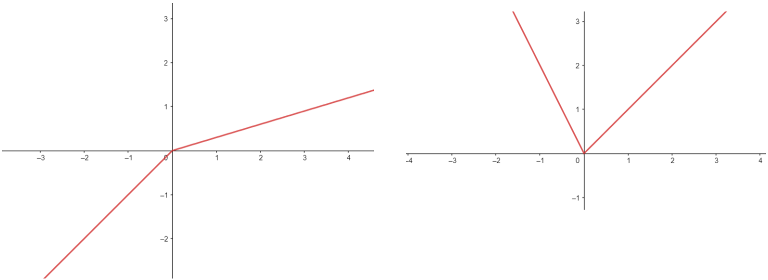

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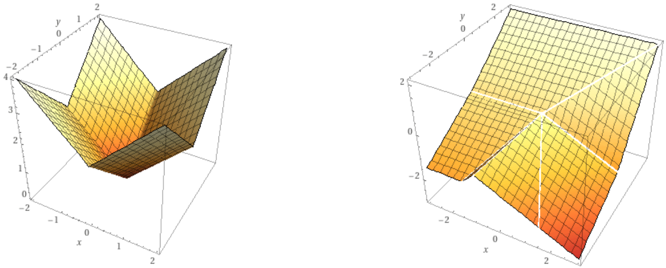

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Theorem (Baker 1968)

- $\mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^\kappa)$ *is isomorphic to the free vector lattice on κ generators.*
- $\mathrm{PWL}_{\mathbb{Z}}(\mathbb{R}^\kappa)$ *is isomorphic to the free ℓ-group on κ generators.*

Piecewise linear functions

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Theorem (Baker 1968)

- *Every κ-generated semisimple vector lattice is isomorphic to $\mathsf{PWL}_{\mathbb{R}}(C)$ where C is a cone that is closed in $\mathbb{R}^\kappa$.*
- *Every κ-generated semisimple ℓ-group is isomorphic to $\mathsf{PWL}_{\mathbb{Z}}(C)$ where C is a cone that is closed in $\mathbb{R}^\kappa$.*

A cone a subset of $\mathbb{R}^\kappa$ closed under multiplication by nonnegative scalars.

Baker-Beynon duality

Theorem (Beynon 1974)

- *The category of finitely generated archimedean vector lattices is dually equivalent to the category of closed cones in* $\mathbb{R}^n$ *for* $n \in \mathbb{N}$ *and piecewise linear maps with real coefficients.*

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- *The category of finitely generated archimedean ℓ-groups is dually equivalent to the category of closed cones in $\mathbb{R}^n$ for $n \in \mathbb{N}$ and piecewise linear maps with integer coefficients.*

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Basic Galois connection (Caramello, Marra, and Spada 2021)

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For any $T \subseteq \mathcal{F}_\kappa$ and $S \subseteq A^\kappa$, we define the following operators.

$$\mathbb{V}_A(T) = \{x \in A^\kappa \mid t(x) = 0 \text{ for all } t \in T\}$$
$$\mathbb{I}_A(S) = \{t \in \mathcal{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\}.$$

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Basic Galois connection

$$T \subseteq \mathbb{I}_A(S) \quad \text{iff} \quad S \subseteq \mathbb{V}_A(T)$$

From a connection to a duality

The key tool: Algebraic Nullstellensatz

- Let I be an ℓ-ideal of $\mathcal{F}_\kappa$. We have $I = \mathbb{I}_A(x)$ for some $x \in A^\kappa$ iff $\mathcal{F}_\kappa/I$ embeds into A.

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Theorem

The Galois connection induces a dual equivalence between

- *the category of algebras of V that are subdirect products of subalgebras of A, and*
- *the category of subsets of type $\mathbb{V}_A(I)$ of A^κ where κ ranges over all the cardinal numbers.*

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$\mathcal{F}_{\kappa} \,/\, \mathbb{I}_{\mathbb{R}}(C) \cong \mathsf{PWL}_{\mathbb{R}}(C)$ (vector lattices)
$\mathcal{F}_{\kappa} \,/\, \mathbb{I}_{\mathbb{R}}(C) \cong \mathsf{PWL}_{\mathbb{Z}}(C)$ (ℓ-groups)

That is, Baker-Beynon duality.

Theorem (Beynon 1974, revisited)

- *The category of semisimple vector lattices is dually equivalent to the category of closed cones in $\mathbb{R}^\kappa$ and piecewise linear maps with real coefficients.*
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How to get such an algebra

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Theorem

Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ such that every κ-generated linearly ordered ℓ-group/vector lattice with $\kappa \leq \gamma$ embeds into $\mathcal{U}$.

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If $\kappa \leq \gamma$, then every κ-generated ℓ-group/vector lattice is subdirect product of totally ordered ones, that are subalgebras of $\mathcal{U}$!

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The subsets of $\mathcal{U}^\kappa$ of type $\mathbb{V}_{\mathcal{U}}(I)$ are the closed set of a Zariski-like topology.

Two remarks:

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- All of this can be done with a generic ℓ-group that embeds all ordered ones (up to a cardinality). With ultrapowers, $\mathbb{I}_{\mathcal{U}}(a)$= prime ℓ-ideal.
- If you add a strong unit everything works. Via Mundici's equivalence, you can work with the equivalent categories of MV-algebras and Riesz MV-algebras, (that are varieties!) and $A = [0, 1]$.

A more concrete view on the duality

Every piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ can be extended to a function ${}^*f : \mathcal{U} \to \mathcal{U}$ by setting ${}^*f([(r_i)_{i \in I}]) = [(f(r_i))_{i \in I}]$.

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$$C\subseteq\mathcal{U}^\kappa\mapsto\mathcal{F}_\kappa/\mathbb{I}_{\mathcal{U}}(C)\quad \mathcal{F}_\kappa/J\mapsto\mathbb{V}_{\mathcal{U}}(J)$$

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We define:

$$ {}^*\mathrm{PWL}_{\mathbb{R}}(\mathcal{U}^\kappa) = \{{}^*f \mid f \in \mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^\kappa)\}, \quad {}^*\mathrm{PWL}_{\mathbb{Z}}(\mathcal{U}^\kappa) = \{{}^*f \mid f \in \mathrm{PWL}_{\mathbb{Z}}(\mathbb{R}^\kappa)\}. $$

If $X \subseteq \mathcal{U}^\kappa$, we can consider ${}^*\mathrm{PWL}_{\mathbb{R}}(X)$ and ${}^*\mathrm{PWL}_{\mathbb{Z}}(X)$.

$$ C \subseteq \mathcal{U}^\kappa \mapsto \mathcal{F}_\kappa \,/\, \mathbb{I}_{\mathcal{U}}(C) \quad \mathcal{F}_\kappa \,/\, J \mapsto \mathbb{V}_{\mathcal{U}}(J) $$

When $C = \mathbb{V}_{\mathcal{U}}(J)$ for some J,

- $\mathcal{F}_\kappa \,/\, \mathbb{I}_{\mathcal{U}}(C) \cong {}^*\mathrm{PWL}_{\mathbb{R}}(C)$ (vector lattices).
- $\mathcal{F}_\kappa \,/\, \mathbb{I}_{\mathcal{U}}(C) \cong {}^*\mathrm{PWL}_{\mathbb{Z}}(C)$ (ℓ-groups).

Correspondence between ℓ-ideals and closed subsets

$\mathcal{F}_{\kappa}$	$\mathbb{R}^{\kappa}$	$\mathcal{U}^{\kappa}$
maximal ℓ-ideals	half-lines from the origin	$\mathbb{V}_{\mathcal{U}}\,\mathbb{I}_{\mathcal{U}}$-closure of standard points (except the origin) $=$ half-lines from the origin through a standard point

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intersections of maximal ℓ-ideals	closed cones	$\mathbb{V}_\mathcal{U}\,\mathbb{I}_\mathcal{U}$-closure of standard subsets

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ℓ-ideals		$\mathbb{V}_\mathcal{U}\,\mathbb{I}_\mathcal{U}$-closed subsets

Thank you!