Modal logic over semi-primal algebras

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Joint work (in progress) with Alexander Kurz and Bruno Teheux

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Modal logic, coalgebraically

Algebraic semantics of classical modal logic can be summarized as follows:

$$\begin{tikzcd} \mathsf{Krip} \arrow[r, bend left] & \mathsf{MA} \arrow[l, bend left] \end{tikzcd}$$

Krip: Kripke frames with bounded morphisms
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Going many-valued:

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Replace the two-element Boolean algebra $\mathbf{2}$ by another finite algebra $\mathbf{L}$.

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Main Assumption

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A finite algebra $\mathbf{P}$ is *primal* if every $f\colon P^k \to P$ is term-definable in $\mathbf{P}$.

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A finite algebra $\mathbf{L}$ is *semi-primal* if every $f\colon L^k \to L$ which preserves subalgebras is term-definable in $\mathbf{L}$.

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Definition (Pixley 1970)

A finite algebra $\mathbf{Q}$ is *quasi-primal* if every $f\colon Q^k \to Q$ which preserves internal isomorphisms is term-definable.
(Quasi-primal algebras are precisely the finite discriminator algebras)

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$$C_n = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$$

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  where $f(0) = 0$, $f(1) = 1$ and $f(\frac{i}{n}) = \frac{i+1}{n}$ for $1 \leq i \leq n-1$

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  where $f(0) = 0$, $f(1) = 1$ and $f(\frac{i}{n}) = \frac{i+1}{n}$ for $1 \leq i \leq n-1$
- $\mathbf{T}_n = (C_n, \wedge, \vee, 0, 1, (T_{\frac{i}{n}})_{i=0}^n)$

Examples 2

Proposition

Let **L** be a finite algebra with bounded lattice reduct. Then **L** is semi-primal if and only if for every $a \in L$ the following $T_a : L \to L$ is term-definable in **L**:

$$T_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a. \end{cases}$$

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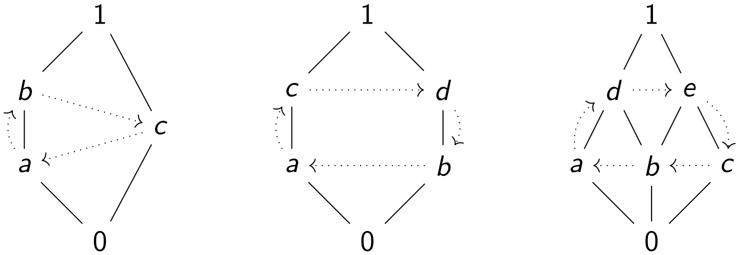
- Therefore, there exists a semi-primal algebra based on *any* bounded lattice.
- Adding the terms T_a is *not the same* as adding constants a.
- Given a finite bounded distributive lattice $\mathbf{D}$, there is an axiomatization of modal logic over $(\mathbf{D}, \to, (T_d)_{d \in D})$ with Heyting implication interpreted on (crisp) Kripke frames. (Maruyama 2009)

Examples 3

Call an algebra $\mathbf{L} = (L, \wedge, \vee, 0, 1,')$ *pseudo-logic* if $0' = 1$ and $1' = 0$.

Examples of semi-primal pseudo-logics: (Davey, Schumann, Werner 1991)



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Reasonable requirements for the algebra of truth-degrees **L**:

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Morphisms : $(X, \mathbf{r}) \to (X', \mathbf{r}')$ are continuous maps $f\colon X \to X'$ with

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Theorem (Keimel, Werner 1974 & Clark, Davey 1998)

There is a dual equivalence

$$\begin{tikzcd} \mathsf{Stone}_{\mathbf{L}} \arrow[r, bend left] & \mathcal{V}\mathbf{L} \arrow[l, bend left] \end{tikzcd}$$

Forgetting topology

To obtain our coalgebraic base category $\mathcal{X}$, we *forget topology*:

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Lifting functors (1)

Proposition

To every Set-endofunctor T there is natural way to associate a $\mathsf{Set}_{\mathbf{L}}$-endofunctor T' with $\mathsf{T}'\mathsf{U} = \mathsf{T}$ (here U is the forgetful functor).

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Example 1: $T = \mathcal{P}$, the (covariant) powerset functor. Then $\mathrm{Coalg}(\mathcal{P}')$ corresponds to the following:

Definition

A *crisp* **L**-*frame* is a triple $(W, R, \mathbf{r})$ such that

- (W, R) is a Kripke-frame.
- $(W, \mathbf{r}) \in \mathsf{Set}_{\mathbf{L}}$ (i.e. $\mathbf{r} : W \to \mathbb{S}(\mathbf{L})$).
- Compatibility: $wRw' \Rightarrow \mathbf{r}(w') \subseteq \mathbf{r}(w)$.

Lifting functors (2)

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Example 2: $T = \mathcal{L}$, given on objects by $\mathcal{L}(X) = L^X$. Then $\mathsf{Coalg}(\mathcal{L}')$ corresponds to the following:

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Abstract coalgebraic modal logic

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Ingredients:

- A (representation of the) functor M' for the 'syntax'.
- A natural transformation $\delta\colon \mathsf{M}'\Pi \Rightarrow \Pi\mathsf{T}'$ for the 'semantics'.

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$$\Pi\mathsf{T}'W \xrightarrow{\Pi\xi} \Pi W$$

Abstract coalgebraic modal logic

$$\begin{tikzcd} \mathsf{Set}_{\mathbf{L}} \arrow[loop left, "\mathsf{T}'"] \arrow[r, bend left, "\Pi"] & \mathcal{V}\mathbf{L} \arrow[l, bend left, "\Sigma"] \arrow[loop right, "\mathsf{M}'"] \end{tikzcd}$$

Ingredients:

- A (representation of the) functor M' for the 'syntax'.
- A natural transformation $\delta\colon \mathsf{M}'\Pi \Rightarrow \Pi\mathsf{T}'$ for the 'semantics'.

From coalgebras to algebras:

$$W \xrightarrow{\quad \xi \quad} \mathsf{T}'W$$

$$\mathsf{M}'\Pi W \xrightarrow{\quad \delta_W \quad} \Pi\mathsf{T}'W \xrightarrow{\quad \Pi\xi \quad} \Pi W$$

Example

Definition

A *crisp* $\mathbf{L}$-*frame* is $(W, R, \mathbf{r})$ such that

- (W, R) is a Kripke-frame.
- $(W, \mathbf{r}) \in \mathrm{Set}_{\mathbf{L}}$ (i.e. $\mathbf{r} : W \to \mathbb{S}(\mathbf{L})$).
- Compatibility: $wRw' \Rightarrow \mathbf{r}(w') \subseteq \mathbf{r}(w)$.

A *crisp* $\mathbf{L}$-*frame* is $(W, R, \mathbf{r}, Val)$ such that, in addition

$$Val \colon W \times \mathrm{Prop} \to L$$

always satisfies $Val(w, p) \in \mathbf{r}(w)$. We extend Val to all modal formulas using the rule

$$Val(w, \Box\varphi) = \bigwedge \{ Val(w', \varphi) \mid wRw' \}$$

Example

$$
\begin{tikzcd}
\mathsf{Set}_{\mathbf{L}} \arrow[loop left, "\mathcal{P}'"] \arrow[r, bend left, "\Pi"] & \mathcal{V}\mathbf{L} \arrow[l, bend left, "\Sigma"] \arrow[loop right, "\mathsf{M}'"]
\end{tikzcd}
$$

Example

$$\begin{tikzcd} \mathsf{Set}_{\mathbf{L}} \arrow[loop left, "\mathcal{P}'"] \arrow[r, bend left, "\Pi"] & \mathcal{V}\mathbf{L} \arrow[l, bend left, "\Sigma"] \arrow[loop right, "\mathsf{M}'"] \end{tikzcd}$$

- M' takes an algebra $\mathbf{A} \in \mathcal{V}\mathbf{L}$ to the free algebra generated by $\{\Box a \mid a \in A\}$ quotiented by the equations

$$\Box 1 \approx 1, \quad \Box(a \wedge b) \approx \Box a \wedge \Box b, \quad \Box \tau_\ell(a) \approx \tau_\ell(\Box a)$$

Example

$$\begin{tikzcd} \mathsf{Set}_{\mathbf{L}} \arrow[loop left, "\mathcal{P}'"] \arrow[r, bend left, "\Pi"] & \mathcal{V}\mathbf{L} \arrow[l, bend left, "\Sigma"] \arrow[loop right, "\mathsf{M}'"] \end{tikzcd}$$

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- $\delta_X \colon \mathsf{M}'\Pi X \to \Pi\mathcal{P}'X$ is determined by

$$\Box f \mapsto (Y \mapsto \bigwedge_{y \in Y} f(y))$$

Example

$$
\begin{tikzcd}
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- Completeness amounts to injectivity of δ.

On the choice of base category

Usually in MV-modal logic, people seem more interested in the correspondence

$$\begin{tikzcd} \mathsf{T} \arrow[loop left] \mathsf{Set} \arrow[r, bend left] & \mathcal{V}\mathbf{L} \arrow[l, bend left] \arrow[loop right, "\mathsf{M}'"] \end{tikzcd}$$

On the choice of base category

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$$\begin{tikzcd}
\mathsf{Set} \arrow[loop left, "\mathsf{T}"] \arrow[rr, bend left] \arrow[dr, bend right, "\mathsf{V}^{\top}"'] & & \mathcal{V}\mathbf{L} \arrow[loop right, "\mathsf{M}'"] \arrow[ll, bend left] \arrow[dl, bend left] \\
& \mathsf{Set}_{\mathbf{L}} \arrow[ul, bend right, "\mathsf{U}"'] \arrow[ur, bend right] \arrow[loop below, "\mathsf{T}'"] &
\end{tikzcd}$$

On the choice of base category

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$$
\begin{tikzcd}
\mathsf{Set} \arrow[loop left, "\mathsf{T}"] \arrow[rr, shift left] \arrow[dr, shift right, "\mathsf{V}^{\top}"'] & & \mathcal{V}\mathbf{L} \arrow[loop right, "\mathsf{M}'"] \arrow[ll, shift left] \arrow[dl, shift left] \\
& \mathsf{Set}_{\mathbf{L}} \arrow[loop below, "\mathsf{T}'"] \arrow[ul, shift right, "\mathsf{U}"'] \arrow[ur, shift left] &
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$$

$\mathsf{V}^{\top} \dashv \mathsf{U}$

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\end{tikzcd}
$$

$\mathsf{V}^{\top} \dashv \mathsf{U}$

$\mathsf{V}^{\top}(X) = (X, \mathbf{r}^{\top})$ assigns $\mathbf{r}^{\top}(x) = \mathbf{L}$ for all $x \in X$.

More connections

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathsf{Set}_{\mathbf{L}} \arrow[rrr, shift left] \arrow[d, bend right=20, "\mathsf{U}"'] \arrow[d, bend left=50, "\mathsf{C}"] \arrow[d, phantom, shift right=8, "\dashv"] \arrow[d, phantom, "\dashv"] \arrow[d, phantom, shift left=8, "\dashv"] & & & \mathcal{V}\mathbf{L} \arrow[lll, shift left] \arrow[d, bend right=20, "\mathfrak{B}"'] \arrow[d, bend left=50, "\mathsf{Q}"] \arrow[d, phantom, shift right=8, "\vdash"] \arrow[d, phantom, "\vdash"] \arrow[d, phantom, shift left=8, "\vdash"] \\
\mathsf{Set} \arrow[u, bend left=50, "\mathsf{V}^{\top}"] \arrow[u, bend right=20, "\mathsf{V}^{\perp}"'] \arrow[rrr, shift left] & & & \mathsf{BA} \arrow[lll, shift left] \arrow[u, bend left=50, "\mathfrak{M}"] \arrow[u, bend right=20, "\mathsf{I}"']
\end{tikzcd}
$$

More connections

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{Set}_{\mathbf{L}} \arrow[loop left, "\mathsf{T}'"] \arrow[rrr, bend left=8] \arrow[d, bend right=50, ""{name=U}] \arrow[d, bend left=50, ""{name=C}] & & & \mathcal{V}\mathbf{L} \arrow[loop right, "\mathsf{M}'"] \arrow[lll, bend left=8] \arrow[d, bend right=50, ""{name=B}] \arrow[d, bend left=50, ""{name=Q}] \\
\mathsf{Set} \arrow[loop left, "\mathsf{T}"] \arrow[rrr, bend left=8] \arrow[u, bend left=70, "\mathsf{V}^{\top}"] \arrow[u, bend right=20, "\dashv \mathsf{U}" {pos=0.5, left}, phantom] \arrow[u, bend right=20, "\dashv \mathsf{V}^{\bot}"'] \arrow[u, phantom, "\dashv \mathsf{C}" {xshift=4em}] & & & \mathsf{BA} \arrow[loop right, "\mathsf{M}"] \arrow[lll, bend left=8] \arrow[u, bend left=70, "\mathfrak{M}"] \arrow[u, phantom, "\vdash \mathfrak{B}" {xshift=-2em}] \arrow[u, bend right=20, "\vdash \mathsf{I}"] \arrow[u, phantom, "\vdash \mathsf{Q}" {xshift=4em}]
\end{tikzcd}
$$

Thanks for your attention!