Subordination Algebras as Semantic Environment of Input/Output Logic

Xiaolong Wang
joint with Andrea De Domenico, Ali Farjami, Krishna Balajirao Manoorkar, Alessandra Palmigiano, Mattia Panettiere

VU Amsterdam

TACL 2022, Coimbra, June 21, 2022

Table of Contents

Introduction

I/O logic is introduced as

1. agency-related notions
2. non-classical

Introduction

Subordination algebras can be defined as tuples $(A, \prec)$ such that A is a Boolean algebra and $\prec$ is a binary relation on A such that the direct (resp. inverse) image of each element $a \in A$ is a filter (resp. an ideal) of A. Subordination algebras are equivalent presentations of some other algebras.

Table of Contents

I/O logic

Definition

An *input/output logic* is a tuple $\mathbb{L} = (\mathcal{L}, N)$ s.t. $\mathcal{L} = (\mathrm{Fm}, \vdash)$ is a (selfextensional) logic, and $N \subseteq \mathrm{Fm} \times \mathrm{Fm}$ is a relation on Fm, is called a normative system.
For any $\Gamma \subseteq \mathrm{Fm}$, let $N(\Gamma) := \{\psi \mid \exists\alpha(\alpha \in \Gamma \;\&\; (\alpha, \psi) \in N)\}$.

I/O logic

Definition (Output operations)

For any input/output logic $\mathbb{L}_i = (\mathcal{L}, N_i)$, and each $1 \leq i \leq 4$,

$$out_i^N(\Gamma) := N_i(\Gamma) = \{\psi \in \mathrm{Fm} \mid \exists\alpha(\alpha \in \Gamma \ \& \ (\alpha, \psi) \in N_i)\}$$

where $N_i \subseteq \mathrm{Fm} \times \mathrm{Fm}$ is the *closure* of N under (i.e. the smallest extension of N satisfying) the inference rules below, as specified in the table.

I/O logic

$$\frac{}{(\top,\top)}\ (\top) \qquad \frac{(\alpha,\varphi) \quad \beta\vdash\alpha}{(\beta,\varphi)}\ (\mathrm{SI}) \qquad \frac{(\alpha,\varphi) \quad \varphi\vdash\psi}{(\alpha,\psi)}\ (\mathrm{WO})$$

$$\frac{(\alpha,\varphi) \quad (\alpha,\psi)}{(\alpha,\varphi\wedge\psi)}\ (\mathrm{AND}) \qquad \frac{(\alpha,\varphi) \quad (\beta,\varphi)}{(\alpha\vee\beta,\varphi)}\ (\mathrm{OR}) \qquad \frac{(\alpha,\varphi) \quad (\alpha\wedge\varphi,\psi)}{(\alpha,\psi)}\ (\mathrm{CT})$$

I/O logic

N_i	Rules
N_1	($\top$), (SI), (WO), (AND)
N_2	($\top$), (SI), (WO), (AND), (OR)
N_3	($\top$), (SI), (WO), (AND), (CT)
N_4	($\top$), (SI), (WO),(AND), (OR), (CT)

Table of Contents

Subordination algebra

Definition ((Proto-)subordination algebra)

A *proto-subordination algebra* is a tuple $\mathbb{S} = (A, \prec)$ such that A is a (possibly bounded) poset (with bottom denoted $\bot$ and top denoted $\top$ when they exist), and $\prec \subseteq A \times A$. A proto-subordination algebra is named as indicated in the left-hand column in the table below when $\prec$ satisfies the properties indicated in the right-hand column.

Subordination algebra

$(\bot)$	$\bot \prec \bot$	$(\top)$	$\top \prec \top$
(OR)	$a \prec x \;\&\; b \prec x \Rightarrow a \vee b \prec x$	(WO)	$b \prec x \leq y \Rightarrow b \prec y$
(AND)	$a \prec x \;\&\; a \prec y \Rightarrow a \prec x \wedge y$	(SI)	$a \leq b \prec x \Rightarrow a \prec x$
(D)	$a \prec c \Rightarrow \exists b(a \prec b \;\&\; b \prec c)$	(S6)	$a \prec b \Rightarrow \neg b \prec \neg a$
(CT)	$a \prec b \;\&\; a \wedge b \prec c \Rightarrow a \prec c$	(T)	$a \prec b \;\&\; b \prec c \Rightarrow a \prec c$

(DD) $a \prec x_1 \;\&\; a \prec x_2 \Rightarrow \exists x(a \prec x \;\&\; x \leq x_1 \;\&\; x \leq x_2)$

(UD) $a_1 \prec x \;\&\; a_2 \prec x \Rightarrow \exists a(a \prec x \;\&\; a_1 \leq a \;\&\; a_2 \leq a)$

(S9) $\exists c(c \prec b \;\&\; x \prec a \vee c) \iff \exists a'\exists b'(a' \prec a \;\&\; b' \prec b \;\&\; x \leq a' \vee b'$

(SL1) $a \prec b \vee c \Rightarrow \exists b'\exists c'(b' \prec b \;\&\; c' \prec c \;\&\; a \prec b' \vee c')$

(SL2) $b \wedge c \prec a \Rightarrow \exists b'\exists c'(b' \prec b \;\&\; c' \prec c \;\&\; b' \wedge c' \prec a)$

Subordination algebra

Name	Properties
$\Diamond$-premonotone	(SI)
$\blacksquare$-premonotone	(WO)
premonotone	(SI) (WO)
$\Diamond$-directed	(WO) (DD)
$\blacksquare$-directed	(SI) (UD)
$\Diamond$-monotone	(WO) (DD) (SI)
$\blacksquare$-monotone	(SI) (UD) (WO)
directed/monotone	(SI) (WO) (UD) (DD)
$\Diamond$-regular	(SI) (WO) (DD) (OR)
$\blacksquare$-regular	(SI) (WO) (UD) (AND)
regular	(SI) (WO) (OR) (AND)
$\Diamond$-normal	(SI) (WO) (DD) (OR) ($\bot$)
$\blacksquare$-normal	(SI) (WO) (UD) (AND) ($\top$)
subordination algebra	(SI) (WO) (OR) (AND) ($\bot$) ($\top$)

Subordination algebra

Definition

A *model* for an input/output logic $\mathbb{L} = (\mathcal{L}, N)$ is a tuple $\mathbb{M} = (\mathbb{S}, h)$ s.t. $\mathbb{S} = (A, \prec)$ is an $\mathsf{Alg}(\mathcal{L})$-based proto-subordination algebra (i.e. $A \in \mathsf{Alg}(\mathcal{L})$), and $h : \mathrm{Fm} \to A$ is a homomorphism s.t. for all $\varphi, \psi \in \mathrm{Fm}$, if $(\varphi, \psi) \in N$, then $h(\varphi) \prec h(\psi)$.

Table of Contents

Canonical extensions and slanted algebras

Definition

Let A be a subposet of a complete lattice A'.

1. The *canonical extension* of a poset A is a complete lattice A^δ containing A as a dense and compact subposet.
2. An element $k \in A'$ is *closed* if $k = \bigwedge F$ for some down-directed $F \subseteq A$; an element $o \in A'$ is *open* if $o = \bigvee I$ for some up-directed $I \subseteq A$;
3. A is *dense* in A' if every element of A' can be expressed both as the join of closed elements and as the meet of open elements of A.
4. A is *compact* in A' if, for all $F, I \subseteq A$ s.t. F is down-directed, I is up-directed, if $\bigwedge F \leq \bigvee I$ then $a \leq b$ for some $a \in F$ and $b \in I$.

The canonical extension A^δ of any poset A always exists and is unique up to an isomorphism fixing A [1]

[1] Dunn, J. M., M. Gehrke and A. Palmigiano, Canonical extensions and relational completeness of some substructural logics, J. Symb. Log. 70 (2005), pp. 713–740

Canonical extensions and slanted algebras

Definition

A *slanted algebra* is a triple $\mathbb{A} = (A, \Diamond, \blacksquare)$ such that A is a poset, and $\Diamond, \blacksquare : A \to A^\delta$ s.t. $\Diamond a \in K(A^\delta)$ and $\blacksquare a \in O(A^\delta)$ for every a. A slanted algebra as above is

1. *tense* if $\Diamond a \leq b$ iff $a \leq \blacksquare b$ for all $a, b \in A$;
2. *monotone* if $\Diamond$ and $\blacksquare$ are monotone;
3. *regular* if $\Diamond$ and $\blacksquare$ are regular (i.e. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ and $\blacksquare(a \wedge b) = \blacksquare a \wedge \blacksquare b$ for all $a, b \in A$);
4. *normal* if $\Diamond$ and $\blacksquare$ are normal (i.e. they are regular and $\Diamond\bot = \bot$ and $\blacksquare\top = \top$).

Canonical extensions and slanted algebras

Definition

For any slanted algebra $\mathbb{A} = (A, \Diamond, \blacksquare)$ the *canonical extension* of $\mathbb{A}$ is the (standard!) modal algebra $\mathbb{A}^\delta := (A^\delta, \Diamond^\sigma, \blacksquare^\pi)$ such that $\Diamond^\sigma, \blacksquare^\pi : A^\delta \to A^\delta$ are defined as follows: for every $k \in K(A^\delta)$, $o \in O(A^\delta)$ and $u \in A^\delta$,

$$\Diamond^\sigma k := \bigwedge\{\Diamond a \mid a \in A \text{ and } k \leq a\} \quad \Diamond^\sigma u := \bigvee\{\Diamond^\sigma k \mid k \in K(A^\delta) \text{ and } k \leq u\}$$

$$\blacksquare^\pi o := \bigvee\{\blacksquare a \mid a \in A \text{ and } a \leq o\}, \quad \blacksquare^\pi u := \bigwedge\{\blacksquare^\pi o \mid o \in O(A^\delta) \text{ and } u \leq o$$

For any slanted algebra $\mathbb{A}^\delta$, any assignment $v : \mathrm{PROP} \to \mathbb{A}$ uniquely extends to a homomorphism $v : \mathcal{L} \to \mathbb{A}^\delta$

Canonical extensions and slanted algebras

Definition

A modal inequality $\phi \leq \psi$ is *satisfied* in a slanted algebra $\mathbb{A}$ under the assignment v (notation: $(\mathbb{A}, v) \models \phi \leq \psi$) if $(\mathbb{A}^\delta, e \cdot v) \models \phi \leq \psi$ in the usual sense, where $e \cdot v$ is the assignment on $\mathbb{A}^\delta$ obtained by composing the canonical embedding $e : \mathbb{A} \to \mathbb{A}^\delta$ to the assignment $v : \mathrm{Prop} \to \mathbb{A}$. Moreover, $\phi \leq \psi$ is *valid* in $\mathbb{A}$ (notation: $\mathbb{A} \models \phi \leq \psi$) if $(\mathbb{A}^\delta, e \cdot v) \models \phi \leq \psi$ for every assignment v into $\mathbb{A}$ (notation: $\mathbb{A}^\delta \models_{\mathbb{A}} \phi \leq \psi$).

Table of Contents

Proto-subordination algebras and slanted algebras

Let $\mathbb{S} = (A, \prec)$ be a proto-subordination algebra s.t. $\mathbb{S} \models \mathrm{(DD)} + \mathrm{(UD)}$. The slanted algebra associated with $\mathbb{S}$ is $\mathbb{S}^* = (A, \Diamond, \blacksquare)$ s.t. $\Diamond a := \bigwedge {\prec}[a]$ and $\blacksquare a := \bigvee {\prec^{-1}}[a]$ for any a. From $\mathbb{S} \models \mathrm{(DD)}$ it follows that ${\prec}[a]$ is down-directed for every $a \in A$, hence $\Diamond a \in K(A^\delta)$. Likewise, $\mathbb{S} \models \mathrm{(UD)}$ guarantees that $\blacksquare a \in O(A^\delta)$ for all $a \in A$.

Proto-subordination algebras and slanted algebras

Lemma

For any proto-subordination algebra $\mathbb{S} = (A, \prec)$ *and all* $a, b \in A$,

1. $a \prec b$ *implies* $\Diamond a \leq b$ *and* $a \leq \blacksquare b$.
2. *if* $\mathbb{S} \models (\mathrm{WO}) + (\mathrm{DD})$, *then* $\Diamond a \leq b$ *iff* $a \prec b$.
3. *if* $\mathbb{S} \models (\mathrm{SI}) + (\mathrm{UD})$, *then* $a \leq \blacksquare b$ *iff* $a \prec b$.

Proto-subordination algebras and slanted algebras

Lemma

For any proto-subordination algebra $\mathbb{S} = (A, \prec)$,

1. *If* $\mathbb{S} \models (\mathrm{WO}) + (\mathrm{DD})$, *then:*
   1. $\mathbb{S} \models (\mathrm{SI})$ *iff* $\Diamond$ *on* $\mathbb{S}^*$ *is monotone.*
   2. $\mathbb{S} \models (\mathrm{OR})$ *iff* $\mathbb{S}^* \models \Diamond(a \vee b) \leq \Diamond a \vee \Diamond b$.
   3. $\mathbb{S} \models (\bot)$ *iff* $\mathbb{S}^* \models \Diamond\bot \leq \bot$.
2. *If* $\mathbb{S} \models (\mathrm{SI}) + (\mathrm{UD})$, *then:*
   1. $\mathbb{S} \models (\mathrm{WO})$ *iff* $\blacksquare$ *on* $\mathbb{S}^*$ *is monotone;*
   2. $\mathbb{S} \models (\mathrm{AND})$ *iff* $\mathbb{S}^* \models \blacksquare a \wedge \blacksquare b \leq \blacksquare(a \wedge b)$;
   3. $\mathbb{S} \models (\top)$ *iff* $\mathbb{S}^* \models \top \leq \blacksquare\top$.

Proto-subordination algebras and slanted algebras

For any proto-subordination algebra $\mathbb{S} = (A, \prec)$,

1. $\mathbb{S} \models \prec \subseteq \leq$ iff $\mathbb{S}^* \models a \leq \Diamond a$ iff $\mathbb{S}^* \models \blacksquare a \leq a$.
2. If $\mathbb{S} \models \mathrm{(WO)} + \mathrm{(DD)}$, then $\mathbb{S} \models \leq \subseteq \prec$ iff $\mathbb{S}^* \models \Diamond a \leq a$;
3. if $\mathbb{S} \models \mathrm{(WO)} + \mathrm{(DD)} + \mathrm{(SI)}$, then
   1. $\mathbb{S} \models \mathrm{(T)}$ iff $\mathbb{S}^* \models \Diamond a \leq \Diamond\Diamond a$.
   2. $\mathbb{S} \models \mathrm{(D)}$ iff $\mathbb{S}^* \models \Diamond\Diamond a \leq \Diamond a$.
4. if $\mathbb{S} \models \mathrm{(WO)} + \mathrm{(DD)} + \mathrm{(SI)}$ and is meet-semilattice based, then
   1. $\mathbb{S} \models \mathrm{(CT)}$ iff $\mathbb{S}^* \models \Diamond a \leq \Diamond(a \wedge \Diamond a)$.
   2. $\mathbb{S} \models \mathrm{(SL2)}$ iff $\mathbb{S}^* \models \Diamond(\Diamond a \wedge \Diamond b) \leq \Diamond(a \wedge b)$.
5. if $\mathbb{S} \models \mathrm{(SI)}$, then $\mathbb{S} \models \mathrm{(CT)}$ implies $\mathbb{S} \models \mathrm{(T)}$.
6. if $\mathbb{S}$ is directed and based on $(A, \neg)$ with $\neg$ antitone, involutive, and (left or right) self-adjoint,
   1. $\mathbb{S} \models \mathrm{(S6)}$ iff $\mathbb{S}^* \models \neg\Diamond a = \blacksquare\neg a$, thus $\blacksquare a := \neg\Diamond\neg a$.
   2. $\mathbb{S} \models \mathrm{(S6)}$ iff $\mathbb{S}^* \models \Diamond\neg a = \neg\blacksquare a$, thus $\Diamond a := \neg\blacksquare\neg a$.
7. If $\mathbb{S} \models \mathrm{(SI)} + \mathrm{(UD)} + \mathrm{(WO)}$ and is join-semilattice based, then
   1. $\mathbb{S} \models (\mathrm{S9} \Rightarrow)$ iff $\mathbb{S}^* \models \blacksquare(a \vee \blacksquare b) \leq \blacksquare a \vee \blacksquare b$.
   2. $\mathbb{S} \models (\mathrm{S9} \Leftarrow)$ iff $\mathbb{S}^* \models \blacksquare a \vee \blacksquare b \leq \blacksquare(a \vee \blacksquare b)$.
   3. $\mathbb{S} \models \mathrm{(SL1)}$ iff $\mathbb{S}^* \models \blacksquare(a \vee b) \leq \blacksquare(\blacksquare a \vee \blacksquare b)$.

Table of Contents

The output operators out_i^N for $1 \leq i \leq 4$ associated with a given input/output logic $\mathbb{L} = (\mathcal{L}, N)$ can be given semantic counterparts in the environment of proto-subordination algebras as follows: for every proto-subordination algebra $\mathbb{S} = (A, \prec)$, we let $\mathbb{S}_i := (A, \prec_i)$ where $\prec_i \subseteq A \times A$ is the smallest extension of $\prec$ which satisfies the properties indicated in the following table:

$\prec_i$	Properties
$\prec_1$	($\top$), (SI), (WO), (AND)
$\prec_2$	($\top$), (SI), (WO), (AND), (OR)
$\prec_3$	($\top$), (SI), (WO), (AND), (CT)
$\prec_4$	($\top$), (SI), (WO), (AND), (OR), (CT)

Then, for each $1 \leq i \leq 4$, if $k \in K(A^\delta)$, then

$$\Diamond_i^\sigma k := \bigwedge\{\prec_i[a] \mid a \in A \text{ and } k \leq a\}$$

encodes the algebraic counterpart of $out_i^N(\Gamma)$ for any $\Gamma \subseteq \mathrm{Fm}$, and the characteristic properties of $\Diamond_i$ for each $1 \leq i \leq 4$ are those identified above.

Application

Celani introduces an expansion of Priestley's duality for bounded distributive lattices to *subordination lattices*, i.e. tuples $\mathbb{S} = (A, \prec)$ such that A is a distributive lattice and $\prec \subseteq A \times A$ is a subordination relation. The dual structure of any subordination lattice $\mathbb{S} = (A, \prec)$ is referred to as the *(Priestley) subordination space* of $\mathbb{S}$, and is defined as $\mathbb{S}_* := (X(A), R_\prec)$, where $X(A)$ is the set of prime filters of A, and $R_\prec \subseteq X(A) \times X(A)$is defined as follows: for all prime filters P, Q of A,

$$(P, Q) \in R_\prec \quad \text{iff} \quad \prec[P] := \{x \in A \mid \exists a(a \in P \ \& \ a \prec x)\} \subseteq Q.$$

Application

Up to isomorphism, we can equivalently define the subordination space of $\mathbb{S}$ as follows:

Definition

The *subordination space* associated with a subordination lattice $\mathbb{S} = (A, \prec)$ is $\mathbb{S}_* := (J^{\infty}(A^{\delta}), R_{\prec})$, where $J^{\infty}(A^{\delta})$ is the set of the completely join-irreducible elements of A^{δ}, and $R_{\prec} \subseteq J^{\infty}(A^{\delta}) \times J^{\infty}(A^{\delta})$ such that $(j, i) \in R_{\prec}$ iff $i \leq \Diamond j$.

Lemma

For any subordination lattice $\mathbb{S} = (A, \prec)$, the subordination spaces $\mathbb{S}_*$ given according to the two definitions above are isomorphic.

Application

Proposition

For any subordination lattice $\mathbb{S}$,

1. $\mathbb{S} \models \prec \subseteq \leq$ *iff* $R_{\prec}$ *is reflexive;*
2. $\mathbb{S} \models \mathrm{(D)}$ *iff* $R_{\prec}$ *is transitive, i.e.* $R_{\prec} \circ R_{\prec} \subseteq R_{\prec}$*;*
3. $\mathbb{S} \models \mathrm{(T)}$ *iff* $R_{\prec}$ *is dense, i.e.* $R_{\prec} \subseteq R_{\prec} \circ R_{\prec}$*;*

Application

Proposition

Let $\mathbb{S} = (A, \prec)$ *be a subordination lattice, and* $(X, R_\prec)$ *be its subordination space, then,*

1. $\mathbb{S} \models (\mathrm{CT})$ *iff*

$$\forall P, Q \in X(PR_\prec Q \implies \exists N, O \in X(P, N \subseteq O \& PR_\prec N \& OR_\prec Q));$$

2. $\mathbb{S} \models (\mathrm{S6})$ *iff*

$$\forall P, Q, N \in X(\exists M \in X(M \subseteq P \cap Q \& MR_\prec N)$$

$$\implies \exists K, L \in X(K \subseteq M \cap L \& LR_\prec Q \& KR_\prec N));$$

Table of Contents

Summary and further work

1. ✓Subordination algebra and slanted algebra
2. ✓Semantic for I/O logic and counterpart of subordination space
3. ? designing scalable I/O reasoners for legal applications.
4. ? dynamic-deontic logic.

Further work

Finally, we hope that the bridge established here can be used to improve mathematical models and methods such as topological, algebraic and duality-theoretic techniques in normative reasoning on one hand, and finding conceptual applications for subordination algebra and related literature on the other hand.