A Galois theory of monoids

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Introduction

categorical Galois theory
central extensions $\overset{?}{\longleftrightarrow}$ categorical approach to monoids

Is there a concept of *centrality* for monoid extensions?

- Already the concept of *extension* is non-trivial and interesting!
- In fact, *special Schreier surjections* (the extensions) have properties that central extensions typically have: they are
  1. pullback-stable,
  2. reflected by pullbacks along regular epimorphisms,
  3. generally not closed under composition.

Are the *special Schreier surjections* central in some Galois theory?

- Almost!

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The **Grothendieck group** adjunction

$$\begin{tikzcd}
\mathrm{Mon} \arrow[r, shift left, "\mathrm{gp}"] \arrow[r, phantom, "\perp"] & \mathrm{Gp} \arrow[l, shift left, "\mathrm{mon}"]
\end{tikzcd}$$

- Gp is not a subvariety of Mon
- *M commutative* monoid (perhaps better known: $\mathbb{Z}$ from $\mathbb{N}$!)

$$\mathrm{gp}(M) = (M \times M)/_{\sim}$$

  where $(m, n) \sim (p, q)$ iff $\exists k\colon m + q + k = p + n + k$
- general case:

$$\mathrm{gp}(M) = \frac{\mathrm{F}(M)}{\mathrm{N}(M)}$$

  $\mathrm{F}(M)$ free group on M, and
  $\mathrm{N}(M) \lhd \mathrm{F}(M)$ generated by words $[m_1][m_2][m_1 m_2]^{-1}$
- elements of $\mathrm{gp}(M)$ look like $\overline{[m_1][m_2]^{-1}[m_3][m_4]^{-1} \cdots [m_n]^{\iota(n)}}$
- unit of the adjunction: $\eta_M\colon M \to \mathrm{gp}(M)\colon m \mapsto \overline{[m]}$
- η_M need not be an injection or a surjection [Mal'tsev, 1937]
  1. $\eta_{\mathbb{N}}\colon \mathbb{N} \to \mathbb{Z}$ is an injection, but
  2. there exist non-trivial M for which $\mathrm{gp}(M) = 0$

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Admissibility [Janelidze, 1990]

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the functor $\mathrm{mon}^M \colon (\mathscr{F} \downarrow \mathrm{gp}(M)) \to (\mathscr{E} \downarrow M)$ is fully faithful $\forall M$.

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 & A \arrow[d, two heads, "f"] \\
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- The proof involves fighting with monoids;
- restricting to CMon and Ab makes things a lot easier.
- gp ⊣ mon is not *semi-left-exact* [Cassidy, Hébert & Kelly, 1985]: we have a counterexample when f or g is not surjective.

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The Galois structure $(\mathrm{Mon}, \mathrm{Gp}, \mathrm{gp}, \mathrm{mon}, \mathscr{E}, \mathscr{F})$, where $\mathscr{E}$ and $\mathscr{F}$ are the classes of surjections in Mon and in Gp, is **admissible**: the functor $\mathrm{mon}^M \colon (\mathscr{F} \downarrow \mathrm{gp}(M)) \to (\mathscr{E} \downarrow M)$ is fully faithful $\forall M$.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, dashed, bend right=60, "\mathrm{mon}^M(\alpha)"'] & A \arrow[d, two heads, "f"] \arrow[dd, dashed, bend left=60, "\alpha"] \\
M \arrow[r, "\eta_M"'] & \mathrm{gp}(M) \\
\cdot \arrow[r] \arrow[u] \arrow[ur, phantom, "\urcorner", very near start] & B \arrow[u, two heads, "g"']
\end{tikzcd}$$

- The proof involves fighting with monoids;
- restricting to CMon and Ab makes things a lot easier.
- gp $\dashv$ mon is not *semi-left-exact* [Cassidy, Hébert & Kelly, 1985]: we have a counterexample when f or g is not surjective.

What are the central extensions? [Janelidze & Kelly, 1994]

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$$\begin{tikzcd} N \arrow[r, tail, shift left, "k"] & X \arrow[l, dashed, shift left, "q"] \arrow[r, two heads, shift left, "f"] & Y \arrow[l, shift left, "s"] \end{tikzcd}$$

(f, s) is a **Schreier split epi** iff $\forall x \in X\ \exists! n \in N \colon x = n \cdot sf(x)$

[Patchkoria, 1998]

- k is split by a function q: take $q(x) = n$.
- The *Split Short Five Lemma* is valid for Schreier split epimorphisms [Bourn, Martins-Ferreira, Montoli & Sobral, 2013].
- Schreier split epimorphisms correspond to actions; an **action** of Y on N is a monoid morphism $\varphi \colon Y \to \mathrm{End}(N)$. We may put $\varphi(y)(n) = {}^{y}n = q(s(y) \cdot n)$; conversely, any action φ gives a Schreier split epimorphism

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Proposition [Bourn, Martins-Ferreira, Montoli & Sobral, 2013]

Special Schreier surjections

1. are stable under products and pullbacks, and
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3. they have a kernel which is a group.

A Schreier split epimorphism need not be a special Schreier surjection.

Tentative proposition

For any split epimorphism (f, s), the following are equivalent:

- i f is a trivial extension;
- ii f is a special Schreier surjection.

Proof (i $\Rightarrow$ ii).

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X \arrow[r, two heads, shift left, "f"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, shift left, "s"] \arrow[d, "\eta_Y"] \\
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(f, s) is a **homogeneous split epi** iff $\forall x \in X\ \exists! n \in N \colon x = n \cdot sf(x)$

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$$\begin{tikzcd} N \arrow[r, tail, shift left, "k"] & X \arrow[l, dashed, shift left, "q"] \arrow[r, two heads, shift left, "f"] & Y \arrow[l, shift left, "s"] \end{tikzcd}$$

(f, s) is a **homogeneous** **split epi** iff $\forall x \in X\ \exists! n \in N\colon x = n \cdot sf(x)$

and $\forall x \in X\ \exists! m \in N\colon x = sf(x) \cdot m$

- The *Split Short Five Lemma* is valid for Schreier split epimorphisms [Bourn, Martins-Ferreira, Montoli & Sobral, 2013].
- k is split by a function q: take $q(x) = n$.
- Schreier split epimorphisms correspond to actions:
  An **action** of Y on N is a monoid morphism $\varphi\colon Y \to \mathrm{End}(N)$
  We may put $\varphi(y)(n) = {}^{y}n = q(s(y) \cdot n)$; $\varphi\colon Y \to \mathrm{Aut}(N)$
  conversely, any action φ gives a Schreier split epimorphism

$$\begin{tikzcd} N \arrow[r, tail, shift left] & N \rtimes_{\varphi} Y \arrow[l, dashed, shift left] \arrow[r, two heads, shift left] & Y. \arrow[l, shift left] \end{tikzcd}$$

A regular epimorphism $g\colon X \to Y$ is a **special** **homogeneous** **surjection** iff (π_1, Δ) is a **homogeneous** split epimorphism:

$$\begin{tikzcd} \mathrm{Eq}(g) \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & X \arrow[l, "\Delta" description] \arrow[r, two heads, "g"] & Y \end{tikzcd}$$

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Proposition [Bourn, Martins-Ferreira, Montoli & Sobral, 2013]

Special Schreier surjections

1. are stable under products and pullbacks, and
2. reflected by pullbacks along regular epimorphisms;
3. they have a kernel which is a group.

A Schreier split epi need not be a special Schreier surjection.

Tentative proposition

For any split epimorphism (f, s), the following are equivalent:

i f is a trivial extension;

ii f is a special Schreier surjection.

Proof (i ⇒ ii).

$$\begin{tikzcd}
X \arrow[r, two heads, shift left, "f"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, shift left, "s"] \arrow[d, "\eta_Y"] \\
\mathrm{gp}(X) \arrow[r, two heads, shift left, "\mathrm{gp}(f)"] & \mathrm{gp}(Y) \arrow[l, shift left, "\mathrm{gp}(s)"]
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Proposition [Bourn, Martins-Ferreira, Montoli & Sobral, 2013]

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- ii f is a special **homogeneous** surjection.

Proof (i $\Rightarrow$ ii).

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N \arrow[r, tail] & N \rtimes_{\overline{\varphi}} \mathrm{gp}(Y) \arrow[r, two heads, shift left] & \mathrm{gp}(Y) \arrow[l, shift left]
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$$\begin{tikzcd}
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What are the central extensions?

Theorem

For any surjection of monoids g, the following are equivalent:

- **i** g is a central extension;
- **ii** g is a normal extension;
- **iii** g is a special homogeneous surjection.

Proof (ii ⇔ iii).

$$\begin{tikzcd}
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g is a normal extension ⇔ π_1 is a trivial extension

⇔ π_1 is a special homogeneous surjection

⇔ g is a special homogeneous surjection □

Corollary

Special homogeneous surjections are reflective amongst regular epimorphisms of commutative monoids with cancellation.

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$$\begin{aligned}
g \text{ is a normal extension} \quad &\Leftrightarrow \quad \pi_1 \text{ is a trivial extension} \\
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Conclusion

We explained that

1. the Grothendieck group adjunction

$$\begin{tikzcd} \mathrm{Mon} \arrow[r, shift left, "\mathrm{gp}"] \arrow[r, phantom, "\perp"] & \mathrm{Gp} \arrow[l, shift left, "\mathrm{mon}"] \end{tikzcd}$$

   is part of an admissible Galois structure;

2. its coverings are precisely the *special homogeneous surjections*, a class of "nice" extensions of monoids.

We still didn't capture *centrality* of monoid extensions via Galois theory:

- What happens when composing this adjunction with abelianisation? What kind of central extensions does the adjunction

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