

# A Galois theory of monoids

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# Introduction

categorical Galois theory  
central extensions  $\longleftrightarrow$  ?  $\longleftrightarrow$  categorical approach to monoids

## Is there a concept of *centrality* for monoid extensions?

- ▶ Already the concept of *extension* is non-trivial and interesting!
- ▶ In fact, *special Schreier surjections* (the extensions) have properties that central extensions typically have: they are
  - 1 pullback-stable,
  - 2 reflected by pullbacks along regular epimorphisms,
  - 3 generally not closed under composition.

## Are the *special Schreier surjections* central in some Galois theory?

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# The Grothendieck group adjunction

$$\text{Mon} \begin{array}{c} \xrightarrow{\text{gp}} \\ \perp \\ \xleftarrow{\text{mon}} \end{array} \text{Gp}$$

- ▶ Gp is not a subvariety of Mon
- ▶  $M$  commutative monoid (perhaps better known:  $\mathbb{Z}$  from  $\mathbb{N}$ !)

$$\text{gp}(M) = (M \times M) / \sim$$

where  $(m, n) \sim (p, q)$  iff  $\exists k: m + q + k = p + n + k$

- ▶ general case:

$$\text{gp}(M) = \frac{F(M)}{N(M)}$$

$F(M)$  free group on  $M$ , and

$N(M) \triangleleft F(M)$  generated by words  $[m_1][m_2][m_1m_2]^{-1}$

- ▶ elements of  $\text{gp}(M)$  look like  $\overline{[m_1][m_2]^{-1}[m_3][m_4]^{-1} \cdots [m_n]^{\iota(n)}}$
- ▶ unit of the adjunction:  $\eta_M: M \rightarrow \text{gp}(M): m \mapsto \overline{[m]}$
- ▶  $\eta_M$  need not be an injection or a surjection [Mal'tsev, 1937]
  - 1  $\eta_{\mathbb{N}}: \mathbb{N} \rightarrow \mathbb{Z}$  is an injection, but
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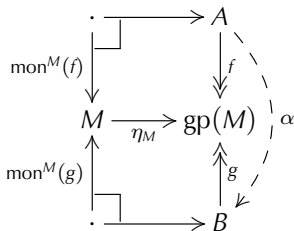
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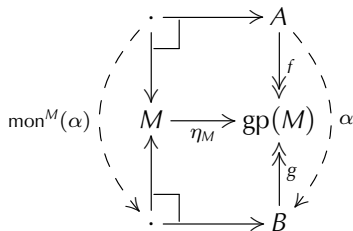


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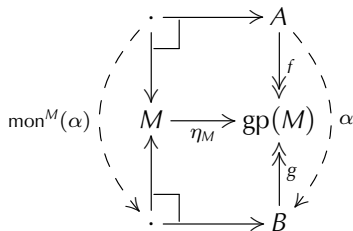


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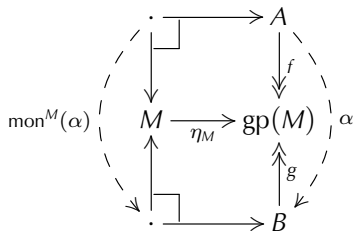
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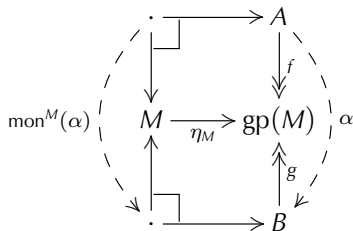


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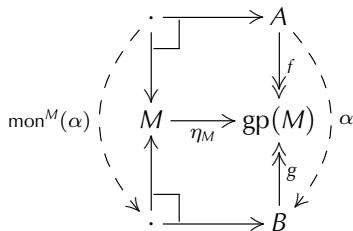


- ▶ The proof involves fighting with monoids;
- ▶ restricting to  $\text{CMon}$  and  $\text{Ab}$  makes things a lot easier.
- ▶  $\text{gp} \dashv \text{mon}$  is not *semi-left-exact* [Cassidy, Hébert & Kelly, 1985]: we have a counterexample when  $f$  or  $g$  is not surjective.

What are the central extensions? [Janelidze & Kelly, 1994]

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**What are the central extensions?** [Janelidze & Kelly, 1994]

## What are the central extensions?

$$N \begin{array}{c} \xrightarrow{k} \\ \xleftarrow{q} \end{array} X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{s} \end{array} Y$$

$(f, s)$  is a **Schreier split epi** iff  $\forall x \in X \exists! n \in N: x = n \cdot sf(x)$

[Patchkoria, 1998]

- ▶  $k$  is split by a function  $q$ : take  $q(x) = n$ .
- ▶ The *Split Short Five Lemma* is valid for Schreier split epimorphisms [Bourn, Martins-Ferreira, Montoli & Sobral, 2013].
- ▶ Schreier split epimorphisms correspond to actions; an **action** of  $Y$  on  $N$  is a monoid morphism  $\varphi: Y \rightarrow \text{End}(N)$ . We may put  $\varphi(y)(n) = {}^y n = q(s(y) \cdot n)$ ; conversely, any action  $\varphi$  gives a Schreier split epimorphism

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A regular epimorphism  $g: X \rightarrow Y$  is a **special Schreier surjection** iff  $(\pi_1, \Delta)$  is a Schreier split epimorphism:

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# What are the central extensions?

Proposition [Bourn, Martins-Ferreira, Montoli & Sobral, 2013]

Special Schreier surjections

- 1 are stable under products and pullbacks, and
- 2 reflected by pullbacks along regular epimorphisms;
- 3 they have a kernel which is a group.

A Schreier split epimorphism need not be a special Schreier surjection.

Tentative proposition

For any split epimorphism  $(f, s)$ , the following are equivalent:

- i  $f$  is a trivial extension;
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Proof (i  $\Rightarrow$  ii).

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$$\text{Eq}(g) \begin{array}{c} \xrightarrow{\pi_1} \\ \xleftarrow{\Delta} \\ \xrightarrow{\pi_2} \end{array} \geq X \xrightarrow{g} \gg Y$$

## What are the central extensions?

$$N \begin{array}{c} \xrightarrow{k} \\ \xleftarrow{q} \end{array} \geq X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{s} \end{array} \gg Y$$

$(f, s)$  is a **homogeneous split epi** iff  $\forall x \in X \exists! n \in N: x = n \cdot sf(x)$   
**and**  $\forall x \in X \exists! m \in N: x = sf(x) \cdot m$

- ▶ The *Split Short Five Lemma* is valid for Schreier split epimorphisms [Bourn, Martins-Ferreira, Montoli & Sobral, 2013].

- ▶  $k$  is split by a function  $q$ : take  $q(x) = n$ .

- ▶ Schreier split epimorphisms correspond to actions:

An **action** of  $Y$  on  $N$  is a monoid morphism  $\varphi: Y \rightarrow \text{End}(N)$

We may put  $\varphi(y)(n) = {}^y n = q(s(y) \cdot n)$ ;  $\varphi: Y \rightarrow \text{Aut}(N)$

conversely, any action  $\varphi$  gives a Schreier split epimorphism

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# What are the central extensions?

Proposition [Bourn, Martins-Ferreira, Montoli & Sobral, 2013]

Special Schreier surjections

- 1 are stable under products and pullbacks, and
- 2 reflected by pullbacks along regular epimorphisms;
- 3 they have a kernel which is a group.

A Schreier split epi need not be a special Schreier surjection.

Tentative proposition

For any split epimorphism  $(f, s)$ , the following are equivalent:

- i  $f$  is a trivial extension;
- ii  $f$  is a special Schreier surjection.

Proof (i  $\Rightarrow$  ii).

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{s} \end{array} & Y \\ \eta_X \downarrow & \lrcorner & \downarrow \eta_Y \\ \text{gp}(X) & \begin{array}{c} \xrightarrow{\text{gp}(f)} \\ \xleftarrow{\text{gp}(s)} \end{array} & \text{gp}(Y) \end{array}$$



# What are the central extensions?

Proposition [Bourn, Martins-Ferreira, Montoli & Sobral, 2013]

Special **homogeneous** surjections

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# What are the central extensions?

## Theorem

For any surjection of monoids  $g$ , the following are equivalent:

- i  $g$  is a central extension;
- ii  $g$  is a normal extension;
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Proof (ii  $\Leftrightarrow$  iii).

$$\text{Eq}(g) \begin{array}{c} \xrightarrow{\pi_1} \\ \xleftarrow{\Delta} \\ \xrightarrow{\pi_2} \end{array} X \xrightarrow{g} Y$$

- $g$  is a normal extension  $\Leftrightarrow \pi_1$  is a trivial extension
- $\Leftrightarrow \pi_1$  is a special homogeneous surjection
- $\Leftrightarrow g$  is a special homogeneous surjection  $\square$

## Corollary

Special homogeneous surjections are reflective amongst regular epimorphisms of commutative monoids with cancellation.

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# Conclusion

We explained that

- 1 the Grothendieck group adjunction

$$\text{Mon} \begin{array}{c} \xrightarrow{\text{gp}} \\ \perp \\ \xleftarrow{\text{mon}} \end{array} \text{Gp}$$

is part of an admissible Galois structure;

- 2 its coverings are precisely the *special homogeneous surjections*, a class of “nice” extensions of monoids.

We still didn't capture *centrality* of monoid extensions via Galois theory:

- ▶ What happens when composing this adjunction with abelianisation?  
What kind of central extensions does the adjunction

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- ▶ Are there other “good” adjunctions?

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