The partial real numbers and the order completion of function rings in pointfree topology

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The aim of this talk is to present the Dedekind order completion of the lattice-ordered ring $\mathrm{C}(L)$ of all continuous real functions on a frame L [4]. Our main tool will be the frame $\mathfrak{L}(\mathbb{IR})$ of *partially defined real numbers* that we will introduce, presented by the same generators of the frame $\mathfrak{L}(\mathbb{R})$ of real numbers [2] but with one relation missing. This is a bigger frame in which $\mathfrak{L}(\mathbb{R})$ embeds canonically. Then $\mathrm{C}(L)$ also embeds canonically in the class $\mathrm{IC}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{IR}), L)$ of *partial real functions* on L. We prove that $\mathrm{IC}(L)$ is Dedekind order complete and describe the Dedekind order completion $\mathrm{C}(L)^{\#}$ of $\mathrm{C}(L)$ inside $\mathrm{IC}(L)$ by

$$\begin{aligned}\mathrm{C}(L)^{\#} = \{h \in \mathrm{IC}(L) \mid &\text{(a) there exist } f, g \in \mathrm{C}(L) \text{ such that } f \leq h \leq g \\ &\text{(b) } h(p,—)^* \leq h(—,q) \text{ and } h(—,q)^* \leq h(p,—) \text{ for any } p < q\}.\end{aligned}$$

As an immediate corollary from this construction we have that for a completely regular L, $\mathrm{C}(L)$ is order complete iff L is extremally disconnected, a result of Banaschewski and Hong [3]. When applied to the spatial case $L = \mathcal{O}X$, our construction improves the construction of Anguelov [1] by avoiding the use of Hausdorff continuous functions.

References

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