Normalizers, centralizers, and action representability in semi-abelian categories

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We investigate the existence of normalizers of subobjects in pointed categories defined in the expected way, as motivated by the standard definition used in the category of groups. We show that, for a semi-abelian category $\mathbb{C}$:

(a) if the category $\mathbb{C}^2$ of morphisms in $\mathbb{C}$ is action representable, then so is the category $\mathbf{Mon}(\mathbb{C})$ of monomorphisms in $\mathbb{C}$;

(b) if $\mathbf{Mon}(\mathbb{C})$ is action representable, then normalizers and centralizers exist in $\mathbb{C}$ and, more generally, in the category of split epimorphisms in $\mathbb{C}$ with any fixed codomain.

On the other hand, assuming that $\mathbb{C}$ is locally well-presentable (that is, locally presentable with filtered colimits commuting with finite limits), we give a necessary and sufficient condition for $\mathbb{C}^2$ to be action representable. In particular, the assumption above holds whenever $\mathbb{C}$ is a variety of universal algebras.