On the algebra of functors valued in a monoidal closed category

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We discuss how the structure of a (symmetric) monoidal closed category $\mathcal{V}$ is reflected in the indexed categories $\mathcal{L}X := \mathcal{V}_0^{X^{\mathrm{op}}}$ and $\mathcal{R}X := \mathcal{V}_0^X$ $(X \in \mathbf{Cat})$. Due to the cartesian structure of $\mathbf{Cat}$, each of them has a monoidal structure inherited pointwise by $\mathcal{V}$ and acts on the opposite of the other one via the internal hom; namely, for $A \in \mathcal{L}X$ and $M \in \mathcal{R}X$ one gets $A \ulcorner^\ell M \in \mathcal{R}X$ by

$$(A \ulcorner^\ell M)x := [Ax, Mx] \qquad ; \qquad (A \ulcorner^\ell M)\alpha := [A\alpha, M\alpha]$$

Substitutions $f^\ell : \mathcal{L}Y \to \mathcal{L}X$ and $f^r : \mathcal{R}Y \to \mathcal{R}X$ along $f : X \to Y$ preserve all the structure and, if $\mathcal{V}$ is complete and cocomplete, the monoidal structures on $\mathcal{L}X$ and $\mathcal{R}X$ are closed and the actions $\ulcorner^\ell$ and $\ulcorner^r$ are biclosed:

$$\frac{\mathcal{L}X(A \otimes B, C)}{\mathcal{L}X(A, B \Rightarrow C)} \qquad ; \qquad \frac{\dfrac{\mathcal{R}X(M, A \ulcorner N)}{\mathcal{R}X(A \odot M, N)}}{\mathcal{L}X(A, M \rhd N)} \tag{1}$$

If $S \in \mathcal{L}X$ is a biaction (that is acts on $\mathcal{V}_0$ by isomorphisms) then

$$\frac{\mathcal{L}X(A \otimes S, B)}{\mathcal{L}X(A, S^{-1} \ulcorner B)} \qquad ; \qquad \frac{\mathcal{L}X(A, M \ulcorner S)}{\mathcal{R}X(M, A \ulcorner S^{-1})}$$

where $S^{-1}\alpha := (S\alpha)^{-1}$, so that $S \Rightarrow B$ has a particularly simple form. It follows that the categories $\mathcal{L}X$ and $\mathcal{R}X$ are enriched in $\mathcal{B}X$ via $\Box^\ell(A \Rightarrow B) \cong \Box^r(A \rhd B)$ and $\Box^r(M \Rightarrow N) \cong \Box^\ell(M \rhd N)$, where $\Box^\ell : \mathcal{L}X \to \mathcal{B}X$ is the coreflection in biactions, as are the adjunctions (1) and $\exists_f^\ell \dashv f^\ell \dashv \forall_f^\ell$.

We show how the situation can be captured by simple axioms, present some consequences and examples and consider the *-autonomous case.

References

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