Combinatorial categories

J. Rosický *

The standard framework for homotopy theory are Quillen model categories where a category $\mathcal{K}$ is equipped with cofibrations, fibrations and weak equivalences. If $\mathcal{K}$ is locally presentable and cofibrations and trivial cofibrations are generated by a set of morphisms then a model category is called combinatorial. There is a more general framework, called cofibration categories, where one has cofibrations and weak equivalences. But one can do homotopy theory based on cofibrations only provided that they form a left part of a weak factorization system. If $\mathcal{K}$ is locally presentable and cofibrations $\mathcal{C}$ are generated by a set of morphisms we say that $(\mathcal{K}, \mathcal{C})$ is a *combinatorial* category. A left adjoint functor between combinatorial categories is called combinatorial if it preserves cofibrations. These functors correspond to left Quillen functors in the model category framework. Our main result (see [3]) is that the (illegitimate) category of combinatorial categories and combinatorial functors is closed under pie limits in CAT. Thus they is closed under pseudolimits and lax limits. A special case is Lurie's lemma (see [1] and [2]) saying that if $\mathcal{K}$ is a combinatorial category and $\mathcal{X}$ a small category then $\mathcal{K}^{\mathcal{X}}$ is combinatorial (with pointwise cofibrations). The key ingredient of the proof is the concept of a *good* colimit (see [1] and [2]) which generalizes transfinite composition from well-ordered chains to well-founded posets. As another application we can mention some results about deconstructible classes where generalized Hill lemma replaces good colimits (see [4]).

References

[1] J. Lurie, *Higher Topos Theory*, Princeton Univ. Press 2009.

[2] M. Makkai, *Rearranging colimits: A categorical lemma due to Jacob Lurie*, www.math.mcgill/makkai

[3] M. Makkai and J. Rosický, *Combinatorial categories*, under preparation.

[4] J. Šťovíček, *Deconstructibility and the Hill lemma in Grothendieck categories*, arXiv:1005.3251.

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