Many for the price of one duality principle for variety-based topological spaces

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In [2], D. Hofmann considered topological spaces as generalized orders, characterizing those ones, which satisfy a suitably defined topological analogue of the complete distributivity law. He showed that the category of distributive spaces is dually equivalent to a certain category of frames, observing that their both represent the idempotent split completion of the same category. The results are based in four submonads of the filter monad $\mathbb{F}$ on the category **Top** of topological spaces [1]. In the talk, we lift the duality of [2] to the setting of lattice-valued topological spaces [3].

Given a variety of algebras $\mathbf{A}$, its reduct $(\mathbf{B}, \|-\|)$, and an $\mathbf{A}$-algebra A, consider the category A-**Top** of A-*topological spaces* (A-*spaces*), whose objects are pairs (X, τ), with X a set and τ an $\mathbf{A}$-subalgebra of the powerset algebra A^X, and whose morphisms $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ are maps $X_1 \xrightarrow{f} X_2$ with $f_A^{\leftarrow}(\alpha) = \alpha \circ f \in \tau_1$ for every $\alpha \in \tau_2$ [6]. There exists a functor $A\text{-}\mathbf{Top} \xrightarrow{\mathcal{O}_A} \mathbf{B}^{op}$, $\mathcal{O}_A((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = \|\tau_1\| \xrightarrow{(f_A^{\leftarrow})^{op}} \|\tau_2\|$, which has a right adjoint [5], thereby providing a monad $\mathbb{T}_A$ on A-**Top**.

Let A-$\mathbf{Top}_0$ be the full subcategory of A-**Top** of T_0 A-*spaces*, i.e., A-spaces (X, τ), where every distinct x_1, $x_2 \in X$ have $\alpha \in \tau$ with $\alpha(x_1) \neq \alpha(x_2)$. There exists the restriction $\mathbb{T}_A^0$ of the monad $\mathbb{T}_A$ to A-$\mathbf{Top}_0$. If $\mathbf{B}$ is enriched in the category **Pos** of posets, one defines a preorder on an A-space (X, τ) by $x_1 \sqsubseteq x_2$ iff $\alpha(x_1) \leqslant \alpha(x_2)$ for every $\alpha \in \tau$, which is an order on T_0 A-spaces (thereby providing a functor A-$\mathbf{Top}_0 \xrightarrow{Spec} \mathbf{Pos}$). For some $\mathbf{A}$ and $\mathbf{B}$, one gets that $\mathbb{T}_A^0$ is of *Kock-Zöberlein type* [1].

Let $(A\text{-}\mathbf{Top}_0)^{\mathbb{T}_A^0}$ be the Eilenberg-Moore category of $\mathbb{T}_A^0$. By [2], a $\mathbb{T}_A^0$-algebra $((X, \tau), h)$ is called $\mathbb{T}_A^0$-*distributive* provided that h has a left adjoint (in the sense of posets) $(X, \tau) \xrightarrow{t} T_A^0(X, \tau)$ in A-$\mathbf{Top}_0$ (which is then a $\mathbb{T}_A^0$-homomorphism with $h \circ t = 1_{(X,\tau)}$). $\mathsf{Spl}(A\text{-}\mathbf{Top}_0)^{\mathbb{T}_A^0}$ is the full subcategory of $(A\text{-}\mathbf{Top}_0)^{\mathbb{T}_A^0}$ of $\mathbb{T}_A^0$-distributive $\mathbb{T}_A^0$-algebras. Moreover, a $\mathbf{B}$-algebra B is called A-*spatial* provided that every $b_1, b_2 \in B$ with $b_1 \nleqslant b_2$ have $p \in \mathbf{B}(B, \|A\|)$ with $p(b_1) \nleqslant p(b_2)$. B is called a $\mathbf{B}$-*frame* provided that it has a $\bigvee$-semilattice reduct, and its primitive operations with non-zero arities distribute over $\bigvee$. $\mathbf{B}$-**Frm** is the full subcategory of $\mathbf{B}$ of A-spatial $\mathbf{B}$-frames.

Following [2], we describe the objects of $\mathsf{Spl}(A\text{-}\mathbf{Top}_A^0)^{\mathbb{T}_A^0}$ and $\mathbf{B}$-**Frm**, and show that the categories are dually equivalent. In particular, one gets the dualities of [2].

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