GEOMETRIC REALIZATIONS OF TRICATEGORIES

BENJAMÍN ALARCÓN HEREDIA
(UNIVERSIDAD DE GRANADA)

arxiv.org 1203.3664

In colaboration with ANTONIO M. CEGARRA

CATEGORIES

For a small category $\mathcal{C}$, its **nerve** $N\mathcal{C}$ is the simplicial set whose p-simplices are p-tuples of composible morphisms in $\mathcal{C}$

$$x_p \longrightarrow x_{p-1} \longrightarrow \dots \longrightarrow x_0$$

CATEGORIES

For a small category $\mathcal{C}$, its **nerve** $N\mathcal{C}$ is the simplicial set whose p-simplices are p-tuples of composible morphisms in $\mathcal{C}$

$$x_p \longrightarrow x_{p-1} \longrightarrow \dots \longrightarrow x_0$$

Taking geometric realization we obtain its **classifying space**

$$B\mathcal{C} = |N\mathcal{C}|$$

BICATEGORIES

For bicategories there are defined **ten** different "nerves":

$$
\begin{tikzcd}
 & & \underline{N}\mathcal{B} \arrow[dl, rightsquigarrow] \arrow[d, rightsquigarrow] \arrow[dr, rightsquigarrow] & & \\
\underline{\underline{\Delta}}\mathcal{B} & \underline{\Delta}^{u}\mathcal{B} \arrow[l] & \underline{S}\mathcal{B} \arrow[l] \arrow[r] & \overline{\nabla}_{u}\mathcal{B} \arrow[r] & \overline{\overline{\nabla}}\mathcal{B} \\
\Delta\mathcal{B} \arrow[u] & \Delta^{u}\mathcal{B} \arrow[l] \arrow[u] & & \nabla_{u}\mathcal{B} \arrow[r] \arrow[u] & \nabla\mathcal{B} \arrow[u]
\end{tikzcd}
$$

BICATEGORIES

For bicategories there are defined **ten** different "nerves":

$$\begin{tikzcd}
 & & \underline{N}\mathcal{B} \arrow[dl, squiggly] \arrow[d, squiggly] \arrow[dr, squiggly] & & \\
\underline{\Delta}\mathcal{B} & \underline{\Delta}^{u}\mathcal{B} \arrow[l] & \underline{S}\mathcal{B} \arrow[l] \arrow[r] & \overline{\nabla}_{u}\mathcal{B} \arrow[r] & \overline{\nabla}\mathcal{B} \\
\Delta\mathcal{B} \arrow[u] & \Delta^{u}\mathcal{B} \arrow[l] \arrow[u] & & \nabla_{u}\mathcal{B} \arrow[u] \arrow[r] & \nabla\mathcal{B} \arrow[u]
\end{tikzcd}$$

But they all produce the same "classifying space" up to homotopy: $\mathbf{B}\mathcal{B}$

(Carrasco, Cegarra, Garzón 2010)

TRICATEGORIES

We studied three "nerves" of a tricategory $\mathcal{T}$

TRICATEGORIES

We studied three "nerves" of a tricategory $\mathcal{T}$

- The Grothendieck nerve $N\mathcal{T}$,

TRICATEGORIES

We studied three "nerves" of a tricategory $\mathcal{T}$

- The Grothendieck nerve $N\mathcal{T}$,
- The Segal nerve $S\mathcal{T}$,

TRICATEGORIES

We studied three "nerves" of a tricategory $\mathcal{T}$

- The Grothendieck nerve $N\mathcal{T}$,
- The Segal nerve $S\mathcal{T}$,
- The Street geometrical nerve $\Delta\mathcal{T}$.

TRICATEGORIES

We studied three "nerves" of a tricategory $\mathcal{T}$

- The Grothendieck nerve $N\mathcal{T}$,
- The Segal nerve $S\mathcal{T}$,
- The Street geometrical nerve $\Delta\mathcal{T}$.

And we proved that

> they all produce the same space up to homotopy

THE GROTHENDIECK NERVE

- It is a pseudo-simplicial bicategory

$$N\mathcal{T} = (N\mathcal{T}, \chi, \omega) : \Delta^{op} \longrightarrow \mathcal{B}icat$$

THE GROTHENDIECK NERVE

- It is a pseudo-simplicial bicategory

$$N\mathcal{T} = (N\mathcal{T}, \chi, \omega) : \Delta^{op} \longrightarrow \mathrm{Bicat}$$

whose bicategory of p-simplices is

$$N_p\mathcal{T} = \bigsqcup_{(x_0,\dots,x_p)\in \mathrm{Ob}\,\mathcal{T}^{p+1}} \mathcal{T}(x_1,x_0) \times \mathcal{T}(x_2,x_1) \times \dots \times \mathcal{T}(x_p,x_{p-1}).$$

THE GROTHENDIECK NERVE

- It is a pseudo-simplicial bicategory

$$N\mathcal{T} = (N\mathcal{T}, \chi, \omega) : \Delta^{op} \longrightarrow \mathbf{Bicat}$$

whose bicategory of p-simplices is

$$N_p \mathcal{T} = \coprod_{(x_0,\dots,x_p)\in \mathrm{Ob}\,\mathcal{T}^{p+1}} \mathcal{T}(x_1,x_0) \times \mathcal{T}(x_2,x_1) \times \dots \times \mathcal{T}(x_p,x_{p-1}).$$

- The morphisms $a^*: N_p\mathcal{T} \longrightarrow N_q\mathcal{T}$ associated to $a:[q] \longrightarrow [p]$ are induced by the unit and composition in $\mathcal{T}$.

THE GROTHENDIECK NERVE

- It is a pseudo-simplicial bicategory

$$N\mathcal{C} = (N\mathcal{C}, \chi, \omega) : \Delta^{op} \longrightarrow \mathbf{Bicat}$$

whose bicategory of p-simplices is

$$N_p\mathcal{C} = \bigsqcup_{(x_0,\dots,x_p)\in \mathrm{Ob}\,\mathcal{C}^{p+1}} \mathcal{C}(x_1,x_0) \times \mathcal{C}(x_2,x_1) \times \dots \times \mathcal{C}(x_p,x_{p-1}).$$

- The morphisms $a^*: N_p\mathcal{C} \longrightarrow N_q\mathcal{C}$ associated to $a:[q] \longrightarrow [p]$ are induced by the unit and composition in $\mathcal{C}$.

- The pseudo-equivalences $\chi_{a,b} : b^* a^* \Rightarrow (ab)^*$ arise from the associativity, left unit and right unit constraints of $\mathcal{C}$

THE GROTHENDIECK NERVE

- It is a pseudo-simplicial bicategory

$$N\mathcal{T} = (N\mathcal{T}, \chi, \omega) : \Delta^{op} \longrightarrow \mathbf{Bicat}$$

whose bicategory of p-simplices is

$$N_p\mathcal{T} = \bigsqcup_{(x_0,\dots,x_p)\in \mathrm{Ob}\,\mathcal{T}^{p+1}} \mathcal{T}(x_1,x_0) \times \mathcal{T}(x_2,x_1) \times \dots \times \mathcal{T}(x_p,x_{p-1}).$$

- The morphisms $a^*: N_p\mathcal{T} \longrightarrow N_q\mathcal{T}$ associated to $a:[q] \longrightarrow [p]$ are induced by the unit and composition in $\mathcal{T}$.

- The pseudo-equivalences $\chi_{a,b}: b^*a^* \Rightarrow (ab)^*$ arise from the associativity, left unit and right unit constraints of $\mathcal{T}$

- The invertible modifications $\omega_{a,b,c}: \chi_{a,bc} \circ \chi_{b,c}a^* \Rrightarrow \chi_{ab,c} \circ c^*\chi_{a,b}$ come from the structure 3-cells π, μ, λ and ρ of $\mathcal{T}$

THE CLASSIFYING SPACE

We can construct a bicategory from the Grothendieck nerve

$$\int_\Delta N\mathcal{T}$$

whose objects are pairs (x, p) with $x = (x_p \to \dots \to x_0)$ an object of $N_p\mathcal{T}$, and whose hom-categories are

$$\int_\Delta N\mathcal{T}\big((y,q),(x,p)\big) = \coprod_{[q]\xrightarrow{a}[p]} N_q\mathcal{T}(y, a^*x).$$

THE CLASSIFYING SPACE

We can construct a bicategory from the Grothendieck nerve

$$\int_\Delta N\mathcal{T}$$

whose objects are pairs (x, p) with $x = (x_p \to \dots \to x_0)$ an object of $N_p\mathcal{T}$, and whose hom-categories are

$$\int_\Delta N\mathcal{T}\big((y,q),(x,p)\big) = \coprod_{[q] \xrightarrow{a} [p]} N_q\mathcal{T}(y, a^*x).$$

Taking nerve again and applying the same construction we obtain a category, whose classifying space will be the classifying space of $\mathcal{T}$, that is

$$B\mathcal{T} = \left| N\left(\int_\Delta N\left(\int_\Delta N\mathcal{T}\right)\right)\right|$$

SOME PROPERTIES

- Trihomomorphisms induce continuous maps.

SOME PROPERTIES

- Trihomomorphisms induce continuous maps.
- Tritransformations induce homotopies

SOME PROPERTIES

- Trihomomorphisms induce continuous maps.
- Tritransformations induce homotopies
- Triequivalences induce homotopy equivalences.

SOME PROPERTIES

- Trihomomorphisms induce continuous maps.
- Tritransformations induce homotopies
- Triequivalences induce homotopy equivalences.
- For any tricategory $\mathcal{T}$ there is an homotopy

$$B1_{\mathcal{T}} \simeq 1_{B\mathcal{T}} : B\mathcal{T} \longrightarrow B\mathcal{T}$$

SOME PROPERTIES

- Trihomomorphisms induce continuous maps.
- Tritransformations induce homotopies
- Triequivalences induce homotopy equivalences.
- For any tricategory $\mathcal{T}$ there is an homotopy

$$B1_{\mathcal{T}} \simeq 1_{B\mathcal{T}} : B\mathcal{T} \longrightarrow B\mathcal{T}$$

- For any trihomomorphisms $H : \mathcal{T} \longrightarrow \mathcal{T}'$ and $H' : \mathcal{T}' \longrightarrow \mathcal{T}''$ there is an homotopy

$$BH'BH \simeq B(H'H) : B\mathcal{T} \longrightarrow B\mathcal{T}''$$

THE SEGAL NERVE

- It is a simplicial bicategory

$$S\mathcal{T} : \Delta^{op} \longrightarrow \mathrm{Bicat}.$$

THE SEGAL NERVE

- It is a simplicial bicategory

$$S\mathcal{T} : \Delta^{op} \longrightarrow Bicat.$$

- The objects of $S_p\mathcal{T}$ are the "normalized" trihomomorphisms

$$[p] \longrightarrow \mathcal{T}.$$

THE SEGAL NERVE

- It is a simplicial bicategory
$$S\mathcal{T} : \Delta^{op} \longrightarrow \mathrm{Bicat}.$$

- The objects of $S_p\mathcal{T}$ are the "normalized" trihomomorphisms
$$[p] \longrightarrow \mathcal{T}.$$

- The 1-cells are the "relative to objects" tritransformations between them.

THE SEGAL NERVE

- It is a simplicial bicategory
$$S\mathcal{T} : \Delta^{op} \longrightarrow \mathrm{Bicat}.$$

- The objects of $S_p\mathcal{T}$ are the "normalized" trihomomorphisms
$$[p] \longrightarrow \mathcal{T}.$$

- The 1-cells are the "relative to objects" tritransformations between them.

- The 2-cells are the "relative to objects" trimodifications between them.

SOME PROPERTIES

- $S\mathcal{T}$ is a special simplical bicategory
  - $S_0\mathcal{T}$ is discrete
  - $S_p\mathcal{T} \longrightarrow S_1\mathcal{T} \,{}_{d_0}\!\times_{d_1} S_1\mathcal{T} \,{}_{d_0}\!\times_{d_1} \cdots {}_{d_0}\!\times_{d_1} S_1\mathcal{T}$ is a biequivalence

SOME PROPERTIES

- $S\mathcal{T}$ is a special simplical bicategory
  - $S_0\mathcal{T}$ is discrete
  - $S_p\mathcal{T} \longrightarrow S_1\mathcal{T} \,{}_{d_0}\!\times_{d_1} S_1\mathcal{T} \,{}_{d_0}\!\times_{d_1} \cdots {}_{d_0}\!\times_{d_1} S_1\mathcal{T}$ is a biequivalence
- There is a pseudo-simplicial homomorphism

$$N\mathcal{T} \longrightarrow S\mathcal{T}$$

such that $N_p\mathcal{T} \longrightarrow S_p\mathcal{T}$ is a biequivalence

SOME PROPERTIES

- $S\mathcal{T}$ is a special simplical bicategory
  - $S_0\mathcal{T}$ is discrete
  - $S_p\mathcal{T} \longrightarrow S_1\mathcal{T} {}_{d_0}\times_{d_1} S_1\mathcal{T} {}_{d_0}\times_{d_1} \cdots {}_{d_0}\times_{d_1} S_1\mathcal{T}$ is a biequivalence
- There is a pseudo-simplicial homomorphism

$$N\mathcal{T} \longrightarrow S\mathcal{T}$$

  such that $N_p\mathcal{T} \longrightarrow S_p\mathcal{T}$ is a biequivalence

- Composing $S\mathcal{T}$ with the classifying space for bicategories we get a simplicial space $BS\mathcal{T} : \Delta^{op} \longrightarrow Top$, and

$$B\mathcal{T} \simeq |BS\mathcal{T}|$$

DELOOPING

- For any monoidal bicategory $(\mathcal{B}, \otimes)$, the loop space $\Omega B(\mathcal{B}, \otimes)$ is a group completion of $B\mathcal{B}$.

DELOOPING

- For any monoidal bicategory $(\mathcal{B}, \otimes)$, the loop space $\Omega B(\mathcal{B}, \otimes)$ is a group completion of $B\mathcal{B}$.

- For any monoidal category $(\mathcal{C}, \otimes)$, the loop space $\Omega B(\mathcal{C}, \otimes)$ is a group completion of $B\mathcal{C}$

(Mac Lane 1965, Stascheff 1963)

DELOOPING

- For any monoidal bicategory $(\mathcal{B}, \otimes)$, the loop space $\Omega B(\mathcal{B}, \otimes)$ is a group completion of $B\mathcal{B}$.

- For any monoidal category $(\mathcal{C}, \otimes)$, the loop space $\Omega B(\mathcal{C}, \otimes)$ is a group completion of $B\mathcal{C}$

  (Mac Lane 1965, Stascheff 1963)

- For any braided monoidal category $(\mathcal{C}, \otimes, c)$, the double loop space $\Omega^2 B(\mathcal{C}, \otimes, c)$ is a group completion of $B\mathcal{C}$.

  (Fiedorowicz 1998, Berger 1997)

THE STREET GEOMETRIC NERVE

- It is a simplicial set

$$\Delta\mathcal{T} : \Delta^{op} \longrightarrow \mathbf{Set}$$

whose p-simplices are the "normal" lax functors

$$[p] \longrightarrow \mathcal{T}.$$

THE STREET GEOMETRIC NERVE

- It is a simplicial set

$$\Delta\mathcal{T} : \Delta^{op} \longrightarrow \mathbf{Set}$$

whose p-simplices are the "normal" lax functors

$$[p] \longrightarrow \mathcal{T}.$$

* The 0-simplices are the objects of $\mathcal{T}$

THE STREET GEOMETRIC NERVE

- It is a simplicial set
$$\Delta\mathcal{T} : \Delta^{op} \longrightarrow \mathbf{Set}$$
whose p-simplices are the "normal" lax functors
$$[p] \longrightarrow \mathcal{T}.$$

* The 0-simplices are the objects of $\mathcal{T}$
* The 1-simplices are the 1-cells of $\mathcal{T}$

THE STREET GEOMETRIC NERVE

- It is a simplicial set

$$\Delta\mathcal{T} : \Delta^{op} \longrightarrow \mathbf{Set}$$

whose p-simplices are the "normal" lax functors

$$[p] \longrightarrow \mathcal{T}.$$

* The 0-simplices are the objects of $\mathcal{T}$
* The 1-simplices are the 1-cells of $\mathcal{T}$
* The 2-simplices are the 2-cells of the form

$$\begin{tikzcd} & F_1 \arrow[dr, bend left] \arrow[d, phantom, "\Downarrow F_{012}"] & \\ F_2 \arrow[ur] \arrow[rr] & {} & F_0 \end{tikzcd}$$

THE STREET GEOMETRIC NERVE

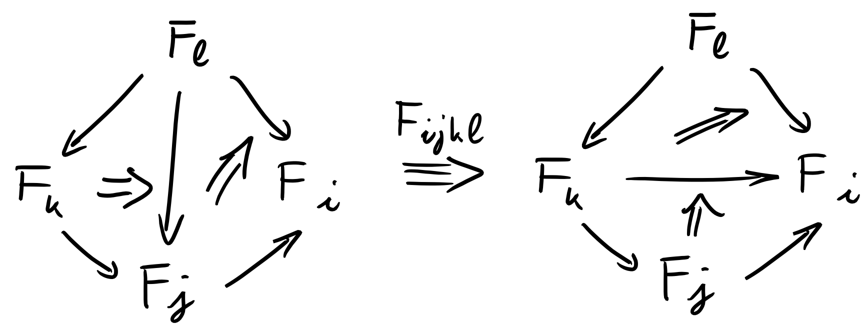
* The p-cells for $p \geq 3$ are collections of 3-cells of the form

$$
\begin{tikzcd}
 & F_l \arrow[dl] \arrow[dd] \arrow[dr] & & & & F_l \arrow[dl] \arrow[dr] & \\
F_k \arrow[dr] \arrow[r, phantom, "\Rightarrow"] & {} \arrow[r, phantom, "\Nearrow"] & F_i \arrow[rr, phantom, "\overset{F_{ijkl}}{\Rrightarrow}"] & & F_k \arrow[rr] \arrow[dr] \arrow[r, phantom, "\Nearrow", near end, shift left=3] & {} \arrow[d, phantom, "\Uparrow"] & F_i \\
 & F_j \arrow[ur] & & & & F_j \arrow[ur] &
\end{tikzcd}
$$

$0 \leq i < j < k < l \leq p$

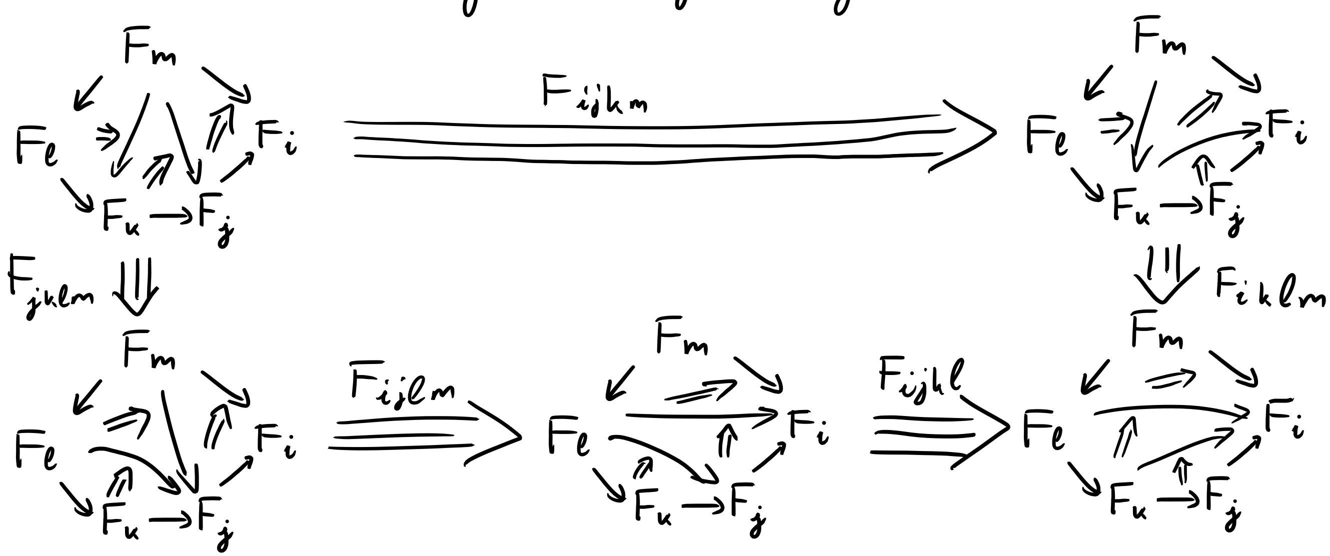
THE STREET GEOMETRIC NERVE

* The p-cells for $p \geq 3$ are collections of 3-cells of the form



$0 \leq i < j < k < l \leq p$

such that the following diagram commutes ($p \geq 4$)



SOME PROPERTIES

- For any tricategory $\mathcal{T}$, $B\mathcal{T} \simeq |\Delta \mathcal{T}|$.

SOME PROPERTIES

- For any tricategory $\mathcal{T}$, $B\mathcal{T} \simeq |\Delta\mathcal{T}|$.
- For any monoidal bicategory $(\mathcal{B}, \otimes)$, $\Delta(\mathcal{B}, \otimes)$ is a Kan complex if and only if $(\mathcal{B}, \otimes)$ is a bicategorical group.

SOME PROPERTIES

- For any tricategory $\mathcal{T}$, $B\mathcal{T} \simeq |\Delta \mathcal{T}|$.
- For any monoidal bicategory $(\mathcal{B}, \otimes)$, $\Delta(\mathcal{B}, \otimes)$ is a Kan complex if and only if $(\mathcal{B}, \otimes)$ is a bicategorical group.
- In that case, the homotopy groups of $\Delta(\mathcal{B}, \otimes)$ can be described using the algebraic structure of $(\mathcal{B}, \otimes)$.

SOME PROPERTIES

- For any tricategory $\mathcal{T}$, $B\mathcal{T} \simeq |\Delta \mathcal{T}|$.
- For any monoidal bicategory $(\mathcal{B}, \otimes)$, $\Delta(\mathcal{B}, \otimes)$ is a Kan complex if and only if $(\mathcal{B}, \otimes)$ is a bicategorical group.
- In that case, the homotopy groups of $\Delta(\mathcal{B}, \otimes)$ can be described using the algebraic structure of $(\mathcal{B}, \otimes)$.

- For any connected CW-complex X with $\pi_i X = 0$ for $i \geq 4$, there is a bicategorical group $(\mathcal{B}, \otimes)$ with an homotopy equivalence

$$B(\mathcal{B}, \otimes) \simeq X$$


THANKS FOR
LISTENING