Asymmetric filter convergence and completeness

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- Asymmetric: quasi-uniform or quasi-nearness structures on biframes
  Technology:
  Paircovers with star refinements: quasi-uniformities
  Paircovers without star refinements: quasi-nearness structures
  Biframes: a pair of subframes which generate the parent frame.

Some technology

Definition

A biframe $L = (L_0, L_1, L_2)$ is a triple, where L_0 is a frame and L_1 and L_2 are subframes of L_0 such that $L_1 \cup L_2$ generates L_0.
A biframe map $h : M \to L$ is a frame map $h : M_0 \to L_0$ such that $h[M_i] \subseteq L_i$, for $i = 1, 2$.

A quasi-nearness on a biframe is a compatible collection of paircovers that forms a filter under meet and refinement. There is no requirement that every uniform paircover has a uniform star-refinement.
(Here a paircover of L is a subset C of $L_1 \times L_2$ such that $\bigvee\{c \wedge \tilde{c} : (c, \tilde{c}) \in C\} = 1$.)

General bifilters

Symmetric case reminder

To discuss questions of convergence and completeness in pointfree topology, one replaces ordinary filters by general ones, as follows:

- A (general) filter $\varphi : L \to T$ is a function between frames that preserves $0, \wedge, 1$.
- The filter φ converges if there is a frame map $h : L \to T$ below φ.
- The filter φ is Cauchy if, for any uniform cover C, one has $\varphi[C]$ a cover.
- The filter φ is regular if $\varphi(a) = \bigvee\{\varphi(z) : z \lhd a\}$.

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- φ preserves $0, \wedge, 1$.
- $\varphi[L_i] \subseteq T_i$ for $i = 1, 2$, that is, φ is part-preserving.
- φ is determined by its action on first and second parts, that is, for any $a \in L_0$, $\varphi(a) = \bigvee\{\varphi(x) \wedge \varphi(y) : x \in L_1, y \in L_2, x \wedge y \leq a\}$.

Definition

Let L be a quasi-nearness biframe and $\varphi : L \to T$ a bifilter.

1. The bifilter $\varphi : L \to T$ converges if there is a biframe map $h : L \to T$ below φ.
2. The bifilter φ is Cauchy if, for any uniform paircover C, $\varphi[C]$ is a paircover.
3. The bifilter φ is regular if $\varphi(a) = \bigvee\{\varphi(z) : z \triangleleft_i a\}$ for $a \in L_i$, $i = 1, 2$.

One can compose bifilters, but it is not (generally) function composition:

Definition

Suppose that $\varphi : L \to M$ and $\rho : M \to T$ are bifilters. We define their composite by $\rho \bullet \varphi \ (a) = \bigvee \left\{ \rho\varphi(x) \wedge \rho\varphi(y) : x \in L_1, y \in L_2, x \wedge y \leq a \right\}$.

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Lemma

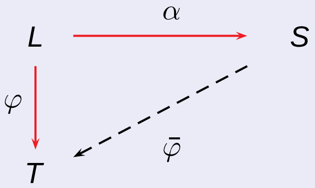
Suppose that $\varphi : L \to M$ and $\rho : M \to T$ are bifilters, with $\rho_{\bullet}\varphi$ as defined above.

1. If ρ is a biframe map, then $\rho_{\bullet}\varphi = \rho\varphi$, since ρ preserves joins.
2. If φ is a uniform map between quasi-nearness biframes and ρ is a Cauchy bifilter, then $\rho_{\bullet}\varphi$ is Cauchy.
3. If φ is a Cauchy bifilter on a quasi-nearness biframe L and ρ is a biframe map, then $\rho_{\bullet}\varphi$ is Cauchy.
4. If φ is a regular bifilter and ρ is a biframe map then $\rho_{\bullet}\varphi$ is regular.

Three universal bifilters

Definition

A bifilter $\alpha : L \to S$ is said to be the universal bifilter on L iff for any bifilter $\varphi : L \to T$ there is a unique biframe map $\bar{\varphi} : S \to T$ such that $\bar{\varphi}\alpha = \varphi$, that is, the following diagram commutes:



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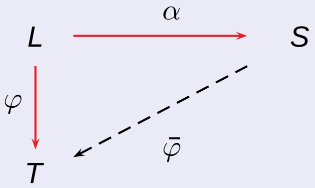

The first universal bifilter

For any biframe L, the right adjoint, s, of the join map $\bigvee : \mathbb{D}L \to L$ is the universal bifilter on L.

We recall the definitions of completeness and completion here:

Definition

Let L be a quasi-nearness biframe.

- We call L quasi-complete if every strict surjection $h : M \to L$ is an isomorphism.
- A quasi-completion of L is a strict surjection $h : M \to L$ where M is quasi-complete.

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There is an immediate link between these notions, in that the right adjoint of any strict surjection is a regular Cauchy bifilter.

- We recall that the quasi-completion CL of L is constructed by forming a quotient of the downset biframe $\mathbb{D}L$. The nucleus n used is the smallest one satisfying these three conditions:

  (N1) For any $U \in (\mathbb{D}L)_0$, $n(U) \subseteq \downarrow(\bigvee U)$.

  (N2) For any $C \in \mathcal{U}L$, $n(\cup \widehat{C}^s) = \downarrow 1$, the top of $(\mathbb{D}L)_0$. Here

$$\begin{aligned} \widehat{C} &= \{(\downarrow c, \downarrow \tilde{c}) : (c, \tilde{c}) \in C\}, \text{ and} \\ \widehat{C}^s &= \{\downarrow c \cap \downarrow \tilde{c} : (c, \tilde{c}) \in C\}. \end{aligned}$$

  (N3) For any $a \in L_i$, $nk_i(a) = \downarrow a$, where

$$k_i(a) = \{x \in L_0 : \exists y \in L_i \text{ with } x \leqslant y \lhd_i a\}.$$

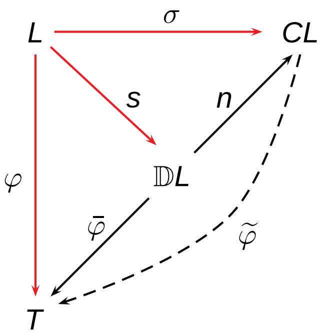
- The congruence on $\mathbb{D}L$ corresponding to the nucleus n is generated by the pairs:
  $\{(\cup \widehat{C}^s, \downarrow 1) : C \in \mathcal{U}L\} \cup$
  $\{(k_1(a), \downarrow a) : a \in L_1\} \cup \{(k_2(b), \downarrow b) : b \in L_2\}$.

The second universal bifilter

Theorem

$\sigma : L \to CL$, the right adjoint of the quasi-completion map, is the universal regular Cauchy bifilter.

PROOF. Let $\varphi : L \to T$ be a regular Cauchy bifilter on L and consider this diagram:



Theorem

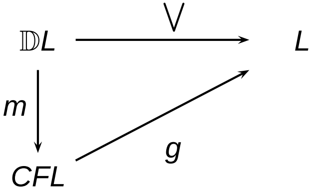
Let L be a quasi-nearness biframe. The following conditions are equivalent:

1. L is quasi-complete.
2. Every regular Cauchy bifilter on L converges.
3. The universal regular Cauchy bifilter $\sigma : L \to CL$ converges.

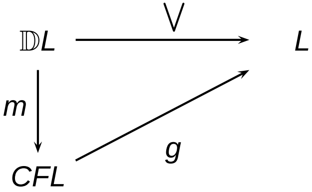
The third universal bifilter

We now construct the Cauchy filter quotient, *CFL*.
This is the quotient of the downset biframe $\mathbb{D}L$ obtained by using the smallest nucleus m on $(\mathbb{D}L)_0$ satisfying the conditions:
(N1) $m(U) \subseteq \downarrow(\bigvee U)$ for all $U \in (\mathbb{D}L)_0$
(N2) $m(\bigcup \widehat{C}^s) = \downarrow 1$ for all $C \in \mathcal{U}L$.
(So no "regularity" condition.)

We note that the congruence on $(\mathbb{D}L)_0$ corresponding to this nucleus is generated by the pairs $\{(\bigcup \widehat{C}^s, \downarrow 1) : C \in \mathcal{U}L\}$.



We define δ to be the right adjoint of g.
Then $\delta : L \to CFL$ turns out to be the universal Cauchy bifilter on L.



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Theorem

Let L be a quasi-nearness biframe. The following conditions are equivalent:

1. The map $g : CFL \to L$ is a biframe isomorphism.
2. Every Cauchy bifilter on L converges.
3. The universal Cauchy bifilter $\delta : L \to CFL$ converges.

So *CL* and *CFL* have several analogous properties. However, they also differ in interesting ways.

- *CF* can be regarded as a functor from quasi-nearness biframes to biframes, where morphisms are dealt with using the following diagram:

$$\begin{array}{ccccc} \mathbb{D}M & \xrightarrow{m_M} & CFM & \xrightarrow{g_M} & M \\ \downarrow{\scriptstyle \mathbb{D}h} & & \downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{D}L & \xrightarrow[m_L]{} & CFL & \xrightarrow[g_L]{} & L \end{array}$$

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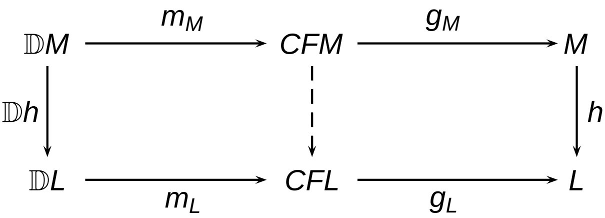

- Is *C*, the quasi-completion, similarly a functor? The obvious modification of this proof cannot be used, as the map in this new argument would not identify the appropriate generating pairs. In fact, it is already known in the setting of nearness frames that the completion is not functorial, so the quasi-completion cannot be functorial either.

The end

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for now....