Tensor products of finitely cocomplete and abelian categories[1]

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[1] With thanks to P. Deligne.

Plan

Deligne’s tensor product of abelian categories

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Deligne's tensor product of abelian categories

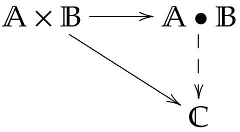
Definition (Deligne)

Given $\mathbb{A}$,$\mathbb{B}$ abelian categories, their tensor product is an abelian category $\mathbb{A} \bullet \mathbb{B}$ with a bilinear *rex* in each variable

$$\mathbb{A} \times \mathbb{B} \to \mathbb{A} \bullet \mathbb{B}$$

that induces equivalences for all abelian $\mathbb{C}$

$$\text{Rex}[\mathbb{A} \bullet \mathbb{B}, \mathbb{C}] \simeq \text{Rex}[\mathbb{A}, \mathbb{B}; \mathbb{C}]$$



Deligne's tensor product of abelian categories

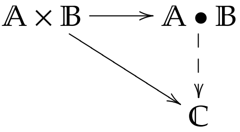
Definition (?,Kelly, well-known)

Given $\mathbb{A}$,$\mathbb{B}$ fin. cocomplete categories, their tensor product is an fin. cocomplete category $\mathbb{A} \boxtimes \mathbb{B}$ with a bilinear *rex* in each variable

$$\mathbb{A} \times \mathbb{B} \to \mathbb{A} \boxtimes \mathbb{B}$$

that induces equivalences for all fin. cocomplete $\mathbb{C}$

$$\mathrm{Rex}[\mathbb{A} \boxtimes \mathbb{B}, \mathbb{C}] \simeq \mathrm{Rex}[\mathbb{A}, \mathbb{B}; \mathbb{C}]$$



Example

For k-algebras R, S,

$$R\text{–}\mathrm{Mod}_f \times S\text{–}\mathrm{Mod}_f \xrightarrow{\otimes_k} R \otimes S\text{–}\mathrm{Mod}_f$$

gives

$$\begin{aligned} R \otimes S\text{–}\mathrm{Mod}_f &\simeq R\text{–}\mathrm{Mod}_f \boxtimes S\text{–}\mathrm{Mod}_f \\ &\simeq R\text{–}\mathrm{Mod}_f \bullet S\text{–}\mathrm{Mod}_f \quad \text{if abelian} \end{aligned}$$

Deligne's tensor product has been used in

- Representations and classification of Hopf algebras.
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Questions

Example (Existence of $\boxtimes$)

For fin. cocomplete $\mathbb{A}$,$\mathbb{B}$, the tensor $\mathbb{A} \boxtimes \mathbb{B}$ exists.

$$\mathbb{A} \boxtimes \mathbb{B} \simeq \mathrm{Lex}[\mathbb{A}^{op}, \mathbb{B}^{op}; k\text{-}\mathrm{Mod}]_f$$

Deligne does not show that his tensor product exists in general. We may ask:

1. Does Deligne's tensor product always exist? No.
2. For fin. cocomplete categories $\mathbb{A}$,$\mathbb{B}$, is $\mathbb{A} \boxtimes \mathbb{B}$ always abelian whenever $\mathbb{A}$,$\mathbb{B}$ are so? No
3. For abelian $\mathbb{A}$,$\mathbb{B}$, their Deligne tensor product $\mathbb{A} \bullet \mathbb{B}$ exists iff $\mathbb{A} \boxtimes \mathbb{B}$ is abelian. Yes

$$2 + 3 \Rightarrow 1$$

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Existence of Deligne's product

Lemma

For abelian $\mathbb{A}$,$\mathbb{B}$, *if* $\mathbb{A} \boxtimes \mathbb{B}$ *is abelian then* $\mathbb{A} \bullet \mathbb{B}$ *exists and is (equivalent to)* $\mathbb{A} \boxtimes \mathbb{B}$.

Proof.

Need $\mathbb{A} \times \mathbb{B} \to \mathbb{A} \boxtimes \mathbb{B}$ to induce

$$\mathrm{Rex}[\mathbb{A} \boxtimes \mathbb{B}, \mathbb{C}] \simeq \mathrm{Rex}[\mathbb{A}, \mathbb{B}; \mathbb{C}]$$

for all **abelian** $\mathbb{C}$. But by definition of $\boxtimes$ this is true for any fin. cocomplete $\mathbb{C}$. □

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Existence of Deligne's product

For a fin. cocomplete $\mathbb{A}$, write $\hat{\mathbb{A}} = \text{Lex}[\mathbb{A}^{op}, k\text{-}\mathrm{Mod}]$

Lemma

If $\mathbb{A} \bullet \mathbb{B}$ exists, then $\widehat{\mathbb{A} \bullet \mathbb{B}}$ is cocomplete abelian and

$$\mathbb{A} \times \mathbb{B} \to \mathbb{A} \bullet \mathbb{B} \to \widehat{\mathbb{A} \bullet \mathbb{B}}$$

induces

$$\text{Cocts}[\widehat{\mathbb{A} \bullet \mathbb{B}}, \mathbb{C}] \simeq \text{Rex}[\mathbb{A}, \mathbb{B}; \mathbb{C}]$$

for all cocomplete abelian $\mathbb{C}$.

Existence of Deligne's product

Theorem

For abelian $\mathbb{A}$,$\mathbb{B}$, TFAE

1. $\mathbb{A} \bullet \mathbb{B}$ *exists.*
2. $\mathbb{A} \boxtimes \mathbb{B}$ *is abelian.*

Proof.

(2⇒1) Lemma.

(1⇒ 2) By Lemma, enough to prove $\widehat{\mathbb{A} \bullet \mathbb{B}} \simeq \widehat{\mathbb{A} \boxtimes \mathbb{B}}$, i.e.,

$$\widehat{\mathbb{A} \boxtimes \mathbb{B}} \simeq \mathrm{Lex}[\mathbb{A}^{op}, \mathbb{B}^{op}; k\text{-}\mathrm{Mod}]$$

has the universal property of $\widehat{\mathbb{A} \bullet \mathbb{B}}$ and it is **abelian.**

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Proof cont.

Theorem

The reflection

$$[(\mathbb{A}\otimes\mathbb{B})^{op}, k\text{-Mod}] \to \mathrm{Lex}[\mathbb{A}^{op}, \mathbb{B}^{op}; k\text{-Mod}]$$

is exact if $\mathbb{A}$,$\mathbb{B}$ *are abelian.*

Proof.

- Follows from: the reflection $[\mathbb{A}^{op}, k\text{-Mod}] \to \mathrm{Lex}[\mathbb{A}^{op}, k\text{-Mod}]$ is *lex*.
- Follows from:

$$\mathrm{Lex}[\mathbb{A}^{op}, k\text{-Mod}] = \mathrm{Sh}(\mathbb{A}, J) \subset [\mathbb{A}^{op}, k\text{-Mod}]$$

J generated by $\{e : A' \to A \text{ epi}\}$ (because $\mathbb{A}$ is abelian).

Proof cont.

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Summary

We showed, for a pair of abelian categories TFAE

- Their Deligne tensor product exists.
- Their tensor as fin. cocomplete categories is abelian.

Counterexample to existence of Deligne's product

Enough to find two abelian $\mathbb{A}$,$\mathbb{B}$ with $\mathbb{A} \boxtimes \mathbb{B}$ *not* abelian.

Definition/Theorem (Chase, Bourbaki, 1960s)

A k-algebra R is *left coherent* iff R–Mod_f is abelian iff every f.g. left ideal is f.p.

Theorem (Soublin, 1968)

There exist two commutative coherent $\mathbb{Q}$-algebras R, S with $R \otimes S$ not coherent.

Proof.

Set $R = \mathbb{Q}[x]$, $S = (\mathbb{Q}^{\mathbb{N}})[[u, t]]$. □

So R-$\mathrm{Mod}_f \boxtimes S$-$\mathrm{Mod}_f$ is not abelian, and the Deligne's tensor R-$\mathrm{Mod}_f \bullet S$-Mod_f does not exist.

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So $R\text{-}\mathrm{Mod}_f \boxtimes S\text{-}\mathrm{Mod}_f$ is not abelian, and the Deligne's tensor $R\text{-}\mathrm{Mod}_f \bullet S\text{-}\mathrm{Mod}_f$ does not exist.

Conclusion

- Deligne's tensor $\mathbb{A} \bullet \mathbb{B}$ does not always exist.
- When $\mathbb{A} \bullet \mathbb{B}$ exists it is (equivalent to) $\mathbb{A} \boxtimes \mathbb{B}$.
- Better use the product of fin. cocomplete categories $\boxtimes$.

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