The ultra-quasi-metrically injective hull of a T_0-ultra-quasi-metric space

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Let X be a set and $u : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of non-negative reals. Then u is an *ultra-quasi-pseudometric* on X if

(i) $u(x, x) = 0$ for all $x \in X$, and

(ii) $u(x, z) \leq \max\{u(x, y), u(y, z)\}$ whenever $x, y, z \in X$.

Note that the so-called *conjugate* u^{-1} of u, where $u^{-1}(x, y) = u(y, x)$ whenever $x, y \in X$, is an ultra-quasi-pseudometric, too.

The set of open balls

$$\{\{y \in X : u(x, y) < \epsilon\} : x \in X, \epsilon > 0\}$$

yields a base for the topology $\tau(u)$ *induced by* u on X.

If u also satisfies the condition

(iii) for any $x, y \in X$, $u(x, y) = 0 = u(y, x)$ implies that $x = y$, then u is called a T_0-*ultra-quasi-metric.*

Observe that then $u^s = u \vee u^{-1}$ is an *ultra-metric* on X.

We next define a canonical T_0-ultra-quasi-metric on $[0, \infty)$.

Example 1 *Let* $X = [0, \infty)$ *be equipped with* $n(x, y) = x$ *if* $x, y \in X$ *and* $x > y$, *and* $n(x, y) = 0$ *if* $x, y \in X$ *and* $x \leq y$.

It is easy to check that (X, n) *is a* T_0-*ultra-quasi-metric space.*

Note also that for $x, y \in [0, \infty)$ *we have* $n^s(x, y) = \max\{x, y\}$ *if* $x \neq y$ *and* $n(x, y) = 0$ *if* $x = y$.

Observe that the ultra-metric n^s *is complete on* $[0, \infty)$ *(compare Example 2 below).*

Furthermore 0 *is the only non-isolated point of* $\tau(n^s)$.

Indeed $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ *is a compact subspace of* $([0, \infty), n^s)$.

In some cases we need to replace $[0, \infty)$ by $[0, \infty]$ (where for an ultra-quasi-pseudometric u attaining the value ∞ the strong triangle inequality (ii) is interpreted in the obvious way).

In such a case we shall speak of an *extended ultra-quasi-pseudometric*.

In the following we sometimes apply concepts from the theory of (ultra-)quasi-pseudometrics to extended (ultra-)quasi-pseudometrics (without changing the usual definitions of these concepts).

A map $f : (X, u) \to (Y, v)$ between two (ultra-)quasi-pseudometric spaces (X, u) and (Y, v) is called *non-expansive* provided that $v(f(x), f(y)) \leq u(x, y)$ whenever $x, y \in X$.

It is called an *isometric map* provided that $v(f(x), f(y)) = u(x, y)$ whenever $x, y \in X$.

Two (ultra-)quasi-pseudometric spaces (X, u) and (Y, v) will be called *isometric* provided that there exists a bijective isometric map $f : (X, u) \to (Y, v)$.

Lemma 1 *Let* $a, b, c \in [0, \infty)$. *Then the following conditions are equivalent:*

(a) $n(a, b) \leq c$.

(b) $a \leq \max\{b, c\}$.

Corollary 1 *Let (X, u) be an ultra-quasi-pseudometric space. Consider $f : X \to [0, \infty)$ and let $x, y \in X$. Then the following are equivalent:*

(a) $n(f(x), f(y)) \leq u(x, y)$;

(b) $f(x) \leq \max\{f(y), u(x, y)\}$.

Corollary 2 *Let (X, u) be an ultra-quasi-pseudometric space.*

(a) Then $f : (X, u) \to ([0, \infty), n)$ is a contracting map if and only if $f(x) \leq \max\{f(y), u(x, y)\}$ whenever $x, y \in X$.

(b) Then $f : (X, u) \to ([0, \infty), n^{-1})$ is a contracting map if and only if $f(x) \leq \max\{f(y), u(y, x)\}$ whenever $x, y \in X$.

Strongly tight function pairs

Definition 1 *Let* (X, u) *be a* T_0*-ultra-quasi-metric space and let* $\mathcal{FP}(X, u)$ *be the set of all pairs* $f = (f_1, f_2)$ *of functions where* $f_i : X \to [0, \infty)$ $(i = 1, 2)$.

For any such pairs (f_1, f_2) *and* (g_1, g_2) *set*

$$N((f_1, f_2), (g_1, g_2)) = \max\{\sup_{x \in X} n(f_1(x), g_1(x)), \sup_{x \in X} n(g_2(x), f_2(x))\}.$$

It is obvious that N *is an extended* T_0*-ultra-quasi-metric on the set* $\mathcal{FP}(X, u)$ *of these function pairs.*

Let (X, u) be a T_0-ultra-quasi-metric space.

We shall say that a pair $f \in \mathcal{FP}(X, u)$ is *strongly tight* if for all $x, y \in X$, we have $u(x, y) \leq \max\{f_2(x), f_1(y)\}$.

The set of all strongly tight function pairs of a T_0-ultra-quasi-metric space (X, u) will be denoted by $\mathcal{UT}(X, u)$.

Lemma 2 *Let* (X, u) *be a* T_0*-ultra-quasi-metric space. For each* $a \in X$, $f_a(x) := (u(a, x), u(x, a))$ *whenever* $x \in X$, *is a strongly tight pair belonging to* $\mathcal{UT}(X, u)$.

Let (X, u) be a T_0-ultra-quasi-metric space.

We say that a function pair $f = (f_1, f_2)$ is *minimal* among the strongly tight pairs on (X, u) if it is a strongly tight pair and if $g = (g_1, g_2)$ is strongly tight on (X, u) and for each $x \in X$, $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$, then $f = g$.

Minimal strongly tight function pairs are also called *extremal strongly tight function pairs*.

By $\nu_q(X, u)$ (or more briefly, $\nu_q(X)$) we shall denote the set of all minimal strongly tight function pairs on (X, u) equipped with the restriction of N to $\nu_q(X)$, which we shall denote again by N.

We note that the restriction of N to $\nu_q(X)$ is indeed a T_0-ultra-quasi-metric on $\nu_q(X,u)$.

In the following we shall call $(\nu_q(X), N)$ the *ultra-quasi-metrically injective hull* of (X,u).

Corollary 3 *Let (X,u) be a T_0-ultra-quasi-metric space. If $f = (f_1, f_2)$ is minimal strongly tight, then*

$$f_1(x) \leq \max\{f_1(y), u(y,x)\}$$

and

$$f_2(x) \leq \max\{f_2(y), u(x,y)\}$$

whenever $x, y \in X$. Thus

$$f_1 : (X,u) \to ([0,\infty), n^{-1})$$

and

$$f_2 : (X,u) \to ([0,\infty), n)$$

are contracting maps.

Lemma 3 *Suppose that* (f_1, f_2) *is a minimal strongly tight pair on a* T_0*-ultra-quasi-metric space* (X, u).

Then $f_2(x) =$

$$\sup\{u(x,y) : y \in X \text{ and } u(x,y) > f_1(y)\}$$

and $f_1(x) =$

$$\sup\{u(y,x) : y \in X \text{ and } u(y,x) > f_2(y)\}$$

whenever $x \in X$.

Lemma 4 *Let* $(f_1, f_2), (g_1, g_2)$ *be minimal strongly tight pairs of functions on a* T_0*-ultra-quasi-metric space* (X, u). *Then*

$$N((f_1, f_2), (g_1, g_2))$$
$$= \sup_{x \in X} n(f_1(x), g_1(x)) = \sup_{x \in X} n(g_2(x), f_2(x)).$$

Corollary 4 *Let* (X, u) *be a* T_0*-ultra-quasi-metric space. Any minimal strongly tight function pair* $f = (f_1, f_2)$ *on* X *satisfies the following conditions:*

$$f_1(x) = \sup_{y \in X} n(u(y,x), f_2(y)) =$$

$$\sup_{y \in X} n(f_1(y), u(x,y))$$

and

$$f_2(x) = \sup_{y \in X} n(u(x,y), f_1(y)) =$$

$$\sup_{y \in X} n(f_2(y), u(y,x))$$

whenever $x \in X$.

Proposition 1 *Let* $f = (f_1, f_2)$ *be a strongly tight function pair on a* T_0*-ultra-quasi-metric space* (X, u) *such that*

$$f_1(x) \leq \max\{f_1(y), u(y, x)\} \text{ and}$$

$$f_2(x) \leq \max\{f_2(y), u(x, y)\}$$

whenever $x, y \in X$.

Furthermore suppose that there is a sequence $(a_n)_{n \in \mathbb{N}}$ *in* X *with*

$$\lim_{n \to \infty} f_1(a_n) = 0$$

and

$$\lim_{n \to \infty} f_2(a_n) = 0.$$

Then f *is a minimal strongly tight pair.*

Envelopes or hulls of T_0-ultra-quasi-metric spaces

Lemma 5 *Let (X, u) be a T_0-ultra-quasi-metric space. For each $a \in X$, the pair f_a belongs to $\nu_q(X, u)$.*

Theorem 1 *Let (X, u) be a T_0-ultra-quasi-metric space.*

For each $f \in \nu_q(X, u)$ and $a \in X$ we have that $N(f, f_a) = f_1(a)$ and $N(f_a, f) = f_2(a)$.

The map $e_X : (X, u) \to (\nu_q(X, u), N)$ defined by $e_X(a) = f_a$ whenever $a \in X$ is an isometric embedding.

Corollary 5 *Let (X, u) be a T_0-ultra-quasi-metric space.*

Then N is indeed a T_0-ultra-quasi-metric on $\nu_q(X)$.

Lemma 6 *Suppose that* (X, u) *is a* T_0*-ultra-quasi-metric space and* $(f_1, f_2) \in \nu_q(X, u)$ *such that* $f_1(a) = 0 = f_2(a)$ *for some* $a \in X$.

Then $(f_1, f_2) = e_X(a)$.

Lemma 7 *Let* (X, u) *be a* T_0*-ultra-quasi-metric space. Then for any* $f, g \in \nu_q(X, u)$ *we have that*

$$N(f, g) = \sup\{u(x_1, x_2) : x_1, x_2 \in X,$$

$$u(x_1, x_2) > f_2(x_1) \textit{ and } u(x_1, x_2) > g_1(x_2)\}.$$

Remark 1 *It follows from the distance formula in Lemma 7 that for any T_0-ultra-quasi-metric space (X, u) the isometric map $e_X : (X, u) \to (\nu_q(X), N)$ has the following tightness property :*

If q is any ultra-quasi-pseudometric on $\nu_q(X, u)$ such that $q \leq N$ and

$$q(e_X(x), e_X(y)) = N(e_X(x), e_X(y))$$

whenever $x, y \in X$, then

$$N(f, g) = q(f, g)$$

whenever $f, g \in \nu_q(X, u)$.

q-spherical completeness

Let (X, u) be an ultra-quasi-pseudometric space and for each $x \in X$ and $r \in [0, \infty)$ let

$$C_u(x, r) = \{y \in X : u(x, y) \leq r\}$$

be the $\tau(u^{-1})$-closed ball of radius r at x.

Lemma 8 *Let (X, u) be an ultra-quasi-pseudometric space.*

Moreover let $x, y \in X$ and $r, s \geq 0$.

Then $C_u(x, r) \cap C_{u^{-1}}(y, s) \neq \emptyset$ if and only if $u(x, y) \leq \max\{r, s\}$.

Definition 2 *Let (X, u) be an ultra-quasi-pseudometric space. Let $(x_i)_{i \in I}$ be a family of points in X and let $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be families of non-negative reals. We say that*

$$(C_u(x_i, r_i), C_{u^{-1}}(x_i, s_i))_{i \in I}$$

has the strong mixed binary intersection property *provided that $u(x_i, x_j) \leq \max\{r_i, s_j\}$ whenever $i, j \in I$.*

We say that (X, u) is q-spherically complete *provided that each family*

$$(C_u(x_i, r_i), C_{u^{-1}}(x_i, s_i))_{i \in I}$$

possessing the strong mixed binary intersection property satisfies

$$\cap_{i \in I}(C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \neq \emptyset.$$

Remark 2 *It is important to note that in Definition 2 we can assume without loss of generality that the points x_i $(i \in I)$ are pairwise distinct.*

Hence that seemingly weaker condition is equivalent to our definition.

Example 2 *The* T_0*-ultra-quasi-metric space* $([0,\infty),n)$ *is q-spherically complete.*

Remark An ultra-metric space (X,m) is called *spherically complete* if for any family $(x_i)_{i\in I}$ of points of X and any family of positive reals $(r_i)_{i\in I}$ such that

$$m(x_i,x_j)\leq \max\{r_i,r_j\}$$

whenever $i,j\in I$ we have that

$$\bigcap_{i\in I} C_m(x_i,r_i)\neq\emptyset.$$

Let (X, m) be an ultra-metric space.

We recall that the ultra-metrically injective hull $(\nu_s(X), E)$ of X is constructed as follows:

Call a function $f : X \to [0, \infty)$ *strongly tight* provided that $m(x, y) \leq \max\{f(x), f(y)\}$ whenever $x, y \in X$.

It is *minimal strongly tight* if it is minimal with respect to the point-wise order on the strongly tight functions on X.

Note that such a function f satisfies

$$f(x) \leq \max\{f(y), m(x, y)\}$$

whenever $x, y \in X$.

Let $\nu_s(X)$ be the set of all minimal strongly tight functions on (X, m) equipped with

$$E(f, g) = \sup_{x \in X} n^s(f(x), g(x))$$

whenever $f, g \in \nu_s(X)$.

Then the ultra-metric space $(\nu_s(X), E)$ yields the ultra-metrically injective hull of (X, m) with isometric embedding $x \mapsto m(x, \cdot)$ where $x \in X$.

Let us observe that there is a different, but equivalent definition of the ultra-metric distance E, namely

$$E(f,g) = \inf_{x \in X} \max\{f(x), g(x)\}$$

whenever $f, g \in \nu_s(X)$ and $f \neq g$.

Proposition 2 *(a) Let (X, u) be an ultra-quasi-pseudometric space.*

Then (X, u) is q-spherically complete if and only if (X, u^{-1}) is q-spherically complete.

(b) Let (X, u) be a T_0-ultra-quasi-metric space.

If (X, u) is q-spherically complete, then (X, u^s) is spherically complete.

As usual, we shall call a quasi-pseudometric space (X, d) *bicomplete* provided that the pseudometric d^s on X is complete.

We recall that each T_0-ultra-quasi-metric space (X, u) has an up-to-isometry unique T_0-ultra-quasi-metric bicompletion $(\widetilde{X}, \widetilde{u})$, in which X is $\tau(\widetilde{u}^s)$-dense.

Proposition 3 *Each q-spherically complete T_0-ultra-quasi-metric space (X, u) is bicomplete.*

A T_0-ultra-quasi-metric space (Y, u_Y) is called *ultra-quasi-metrically injective* provided that for any T_0-ultra-quasi-metric space (X, u_X), any subspace A of (X, u_X) and any non-expansive map $f : A \to (Y, u_Y)$, f can be extended to a non-expansive map $g : (X, u_X) \to (Y, u_Y)$.

Theorem 2 *A T_0-ultra-quasi-metric space is q-spherically complete if and only if it is ultra-quasi-metrically injective.*

Proposition 4 *Let* (X, u) *be a* T_0*-ultra-quasi-metric space. Then* $(f_1, f_2) \in \nu_q(X, u)$ *implies that* $(f_2, f_1) \in \nu_q(X, u^{-1})$.

It follows that

$s : (\nu_q(X, u), N) \to (\nu_q(X, u^{-1}), N^{-1})$

where s *is defined by* $s((f, g)) = (g, f)$ *whenever* $(f, g) \in \nu_q(X, u)$ *is a bijective isometric map.*

(Indeed the ultra-quasi-metrically injective hull $(\nu_q(X, u), N)$ *of* (X, u) *is isometric to the conjugate space of the ultra-quasi-metrically injective hull*

$$(\nu_q(X, u^{-1}), N)$$

of (X, u^{-1})*.)*

Proposition 5 *Let* (X, m) *be an ultrametric space.*

Then $p(f) = (f, f)$ *defines an isometric embedding of*

$$(\nu_s(X, m), E)$$

into

$$(\nu_q(X, m), N).$$

Proposition 6 *Let* (X, u) *be a* T_0*-ultra-quasi-metric space.*

If $s = (s_1, s_2)$ *is a minimal strongly tight pair of functions on the* T_0*-ultra-quasi-metric space* $(\nu_q(X), N)$*, then*

$$s \circ e_X$$

is a minimal strongly tight pair of functions on (X, u)*.*

Lemma 9 *Let A be a nonempty subset of a T_0-ultra-quasi-metric space (X, u) and let*

$$(r_1, r_2) : A \longrightarrow [0, \infty)$$

be such that for all $x, y \in A$, $u(x, y) \leq \max\{r_2(x), r_1(y)\}$.

Then there exists $(R_1, R_2) : X \longrightarrow [0, \infty)$ which extends the pair (r_1, r_2) such that for all

$$x, y \in X,\ u(x, y) \leq \max\{R_2(x), R_1(y)\}.$$

Moreover, there exists a minimal strongly tight pair (f_1, f_2) of functions defined on X such that for all $x \in X$, $f_1(x) \leq R_1(x)$ and $f_2(x) \leq R_2(x)$.

Proposition 7 *The following statements are true for any T_0-ultra-quasi-metric space (X, u).*

(a) $(\nu_q(X), N)$ is q-spherically complete.

(b) $(\nu_q(X), N)$ is an ultra-quasi-metrically

injective hull of X, i.e. no proper subset of $\nu_q(X)$ which contains X as a subspace is q-spherically complete.

The ultra-quasi-metrically injective hull of the T_0-ultra-quasi-metric space (X, u) is unique up to isometry.

Corollary 6 *The following statements are equivalent for a T_0-ultra-quasi-metric space (X, u) :*

(a) (X, u) is q-spherically complete.

(b) For each $f \in \nu_q(X)$ there is $x \in X$ such that $f_1 = (f_x)_1$ and $f_2 = (f_x)_2$.

(c) For each $f \in \nu_q(X)$ there is $x \in X$ such that $f_1(x) = 0 = f_2(x)$.

Remark 3 *Let* (X, u) *be a* T_0*-ultra-quasi-metric space and let* $\nu_q(X, u)$ *be its ultra-quasi-metrically injective hull.*

Since $\nu_q(X, u)$ *is bicomplete, the* $\tau(N^s)$*-closure of* $e_X(X)$ *in* $\nu_q(X, u)$ *yields a subspace of* $\nu_q(X, u)$ *that is isometric to the (quasi-metric) bicompletion of* (X, u).

Of course, $f \in \nu_q(X, u)$ *belongs to the* $\tau(N^s)$*-closure of* $e_X(X)$ *if and only if there is a sequence* $(a_n)_{n\in\mathbb{N}}$ *in* X *such that* $\lim_{n\to\infty} N^s(f_{a_n}, f) = 0$.

In the light of the distance formula proved above, this statement is equivalent to the existence of a sequence $(a_n)_{n\in\mathbb{N}}$ *in* X *such that* $\lim_{n\to\infty} f_1(a_n) = 0$ *and* $\lim_{n\to\infty} f_2(a_n) = 0$.

Total boundedness in T_0-ultra-quasi-metric spaces

Recall that a quasi-pseudometric space (X, d) is called *totally bounded* provided that the pseudometric space (X, d^s) is totally bounded.

Lemma 10 *Let (X, u) be a T_0-ultra-quasi-metric space that is totally bounded and let $\epsilon > 0$.*

Then there is a finite subset E of X such that

$$\{f_1(x) : f \in \nu_q(X), x \in X, f_1(x) > \epsilon\} \cup$$
$$\{f_2(x) : f \in \nu_q(X), x \in X, f_2(x) > \epsilon\} =$$
$$\{u(e, e') : e, e' \in E, u(e, e') > \epsilon\}.$$

It is known that each totally bounded T_0-quasi-metric space (X, d) has a totally bounded Isbell-hull $\epsilon_q(X, d)$. Next we establish a similar result for T_0-ultra-quasi-metric spaces.

Proposition 8 *If (X, u) is a totally bounded T_0-ultra-quasi-metric space, then the T_0-ultra-quasi-metric space $(\nu_q(X, u), N)$ is totally bounded, too.*

Recall that a compact ultra-metric space (X, m) is spherically complete.

Corollary 7 *Let (X, m) be a totally bounded ultra-metric space. Then the completion of (X, m) is isometric to $(\nu_s(X), E)$.*

As usual, we shall call an ultra-quasi-pseudometric space (X, u) *joincompact* if $\tau(u^s)$ is compact. It is readily seen that a joincompact T_0-ultra-quasi-metric space need not be q-spherically complete.

Example 3 *Let $X = \{0, 1\}$ be equipped with the discrete metric u defined by $u(x, y) = 1$ if $x \neq y$, and $u(x, y) = 0$ otherwise. Then (X, u) is not q-spherically complete, although it is spherically complete.*

We now compute the ultra-quasi-metrically injective hull of (X, u). If $f = (f_1, f_2) \in \nu_q(X)$ is strongly tight, then we have $1 = u(0,1) \leq \max\{f_2(0), f_1(1)\}$ and $1 = u(1,0) \leq \max\{f_2(1), f_1(0)\}$.

If f is also minimal strongly tight, then we only find four pairs

$$((f_1(0), f_1(1)), (f_2(0), f_2(1)))$$

determined as follows:

$$((0,1),(0,1)), ((1,1),(0,0)),$$

$$((0,0),(1,1)), ((1,0),(1,0)).$$

Identifying these points $f = (f_1, f_2)$ according to $(f_1(0), f_1(1)) = (\alpha, \beta)$ with $\alpha, \beta \in \{0, 1\}$

we obtain

$$N((\alpha, \beta), (\alpha', \beta')) = 1$$

if ($\alpha = 1$ and $\alpha' = 0$) or ($\beta = 1$ and $\beta' = 0$), and

$$N((\alpha, \beta), (\alpha', \beta')) = 0$$

otherwise.

In particular the example shows that a spherically complete ultra-metric space need not be q-spherically complete.

Corollary 8 *If* (X, u) *is a* T_0*-ultra-quasi-metric space such that* $\tau(u^s)$ *is compact, then* N^s *induces a compact topology on* $\nu_q(X, u)$.

Lemma 11 *Let* (X, u) *be a* T_0*-ultra-quasi-metric space. Let* $f = (f_1, f_2) \in \nu_q(X)$ *be such that there is* $a \in X$ *with* $f_1(a) \leq \inf_{x \in X} f_2(x)$. *Then* $f_1(a) = 0$.

(Note that the result remains true if f_1 *and* f_2 *are interchanged in the statement.)*

Lemma 12 *Let* (X, u) *be a joincompact* T_0*-ultra-quasi-metric space and let* $f = (f_1, f_2) \in \nu_q(X)$*. Then there is* $x \in X$ *such that* $f_1(x) = 0$ *or* $f_2(x) = 0$.

We note that in the case of an ultrametric Lemma 12 implies the afore-mentioned result that a compact ultra-metric space (X, m) is spherically complete, since all functions $f \in \nu_s(X)$ must be of the form $m(x, \cdot)$ for some $x \in X$ because they have a zero (compare Lemma 6).

On the other hand Example 3 yields two function pairs $((1, 1), (0, 0))$ and $((0, 0), (1, 1))$ witnessing that joincompactness does not imply q-spherical completeness, since there is no $x \in X$ such that $f_1(x) = 0 = f_2(x)$.

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