On some mysterious Mal′tsev conditions and the associated imaginary co-operations

dedicated to George Janelidze

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joint work with Diana Rodelo

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Workshop on Category Theory
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Some mysterious Mal'tsev conditions

Theorem [Hagemann & Mitschke, *On n-permutable congruences*, 1973]

For any equational class $\mathcal{V}$ and any $A \in \mathcal{V}$, the following are equivalent:

1. the congruence relations on A are n-permutable;
2. every reflexive relation R on A satisfies $R^{\mathrm{op}} \leqslant R^{n-1}$;
3. every reflexive relation R on A satisfies $R^{n} \leqslant R^{n-1}$. □

The mystery

- Conditions **2** and **3** do not appear in [Carboni, Kelly & Pedicchio, *Some remarks on Maltsev and Goursat categories*, 1993]

  Nevertheless, all three conditions are purely categorical!
- We could, however, not find a categorical argument, and
- the proof Hagemann and Mitschke refer to was never published: [Hagemann, *Grundlagen der allgemeinen topologischen Algebra*, in preparation]

What's going on?

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The associated imaginary co-operations

Hagemann and Mitschke's result is correct

- **1** $\Leftrightarrow$ **2** is treated in [Martins–Ferreira & VdL, 2010]
  2 $\Leftrightarrow$ **3** is also true for varieties

But what about general categories?

- the result holds in regular categories with finite sums
- proof technique mimics the varietal proof,
- based on Dominique Bourn and Zurab Janelidze's *approximate* or *imaginary* co-operations
  [Bourn & Janelidze, *Approximate Mal'tsev operations*, 2008]

Basic idea [Bourn & Janelidze, 2008]

A Mal'tsev theory contains a *Mal'tsev term* $p(x, y, z)$.

A regular Mal'tsev category has *approximate Mal'tsev co-operations*

$$X \overset{\alpha_X}{\twoheadleftarrow} A(X) \xrightarrow{p_X} X + X + X$$

which may be considered as *imaginary co-operations* $p_X : X \rightsquigarrow 3X$.

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"Whatever can be said about varieties can be proved categorically"

[Hans-E. Porst, yesterday]

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1 Mal'tsev conditions
- The Mal'tsev case: 2-permutability
- The Goursat case: 3-permutability
- *n*-permutable categories

2 Imaginary co-operations
- Approximate Mal'tsev co-operations
- Approximate Goursat co-operations
- Main theorem: *n*-permutability

3 Conclusion

4 Further questions

The Mal'tsev case: 2-permutability

Theorem [Mal'tsev, 1954]

For any variety of algebras $\mathcal{V}$, the following are equivalent:

1. 2-permutability of congruences: $RS = SR$
2. existence of a ternary operation p satisfying $\begin{cases} p(x,y,y) = x \\ p(x,x,y) = y \end{cases}$ $\square$

Such a $\mathcal{V}$ is called a **Mal'tsev variety**.

Theorem [Meisen, 1974; Faro, 1989; Carboni, Lambek & Pedicchio, 1990]

For any regular category $\mathcal{A}$, the following are equivalent:

1. 2-permutability of congruences: $RS = SR$
2. every reflexive relation R is symmetric: $R^{\mathrm{op}} \leqslant R$;
3. every reflexive relation R is transitive: $R^2 \leqslant R$. $\square$

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Theorem [Meisen, 1974; Faro, 1989; Carboni, Lambek & Pedicchio, 1990]

For any regular category $\mathcal{A}$, the following are equivalent:

1. 2-permutability of congruences: $RS = SR$
2. every reflexive relation R is symmetric: $R^{\mathrm{op}} \leqslant R$; $\quad R^{\mathrm{op}} \leqslant R^{n-1}$
3. every reflexive relation R is transitive: $R^2 \leqslant R$. $\quad R^n \leqslant R^{n-1}$ $\square$

Such an $\mathcal{A}$ is called a (regular) **Mal'tsev category**.

The Goursat case: 3-permutability

Theorem [Schmidt, 1969; Grötzer, Wille, 1970; Hagemann & Mitschke, 1973]

For any variety of algebras $\mathcal{V}$, the following are equivalent:

1. 3-permutability of congruences: $RSR = SRS$;
2. existence of quaternary operations p and q satisfying

   $$p(x,y,y,z) = x, \qquad p(x,x,y,y) = q(x,x,y,y), \qquad q(x,y,y,z) = z;$$

3. existence of ternary operations r and s satisfying

   $$r(x,y,y) = x, \qquad r(x,x,y) = s(x,y,y), \qquad s(x,x,y) = y;$$

4. every reflexive relation R satisfies $R^{\mathrm{op}} \leqslant R^2$;
5. every reflexive relation R satisfies $R^3 \leqslant R^2$. □

Such a $\mathcal{V}$ is called a 3**-permutable** or **Goursat** variety.

A regular category with 3-permutable congruences
is called a (regular) **Goursat category**
[Carboni, Lambek & Pedicchio, 1990; Carboni, Kelly & Pedicchio, 1993].

The Goursat case: 3-permutability $\quad n = 3$

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n-permutable categories

Theorem [Schmidt, 1969; Grötzer, Wille, 1970; Hagemann & Mitschke, 1973]

$\mathcal{V}$ is n-**permutable** when the following equivalent conditions hold:

1. n-permutability of congruences: $\overbrace{RSRS\cdots}^{n} = \overbrace{SRSR\cdots}^{n}$;
2. existence of $(n+1)$-ary operations v_0, ..., v_n satisfying

$$\begin{cases} v_0(x_0,\ldots,x_n)=x_0, & v_n(x_0,\ldots,x_n)=x_n, \\ v_{i-1}(x_0,x_0,x_2,x_2,\ldots)=v_i(x_0,x_0,x_2,x_2,\ldots), & i \text{ even}, \\ v_{i-1}(x_0,x_1,x_1,x_3,x_3,\ldots)=v_i(x_0,x_1,x_1,x_3,x_3,\ldots), & i \text{ odd}; \end{cases}$$

3. existence of ternary operations w_1, ..., w_{n-1} satisfying

$$\begin{cases} w_1(x,y,y)=x, & w_{n-1}(x,x,y)=y, \\ w_i(x,x,y)=w_{i+1}(x,y,y), & \text{for } i\in\{1,\ldots,n-2\}; \end{cases}$$

4. every reflexive relation R satisfies $R^{\mathrm{op}} \leqslant R^{n-1}$;
5. every reflexive relation R satisfies $R^n \leqslant R^{n-1}$. □

Notion of n-**permutable category** [Carboni, Kelly & Pedicchio, 1993].

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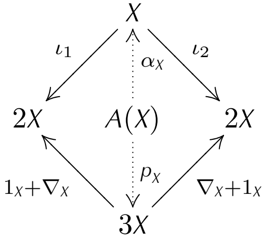
Approximate Mal'tsev co-operations

Natural **approximate Mal'tsev co-operation** on $\mathcal{A}$:



$$\begin{cases} \left\langle \begin{smallmatrix} x \\ x \\ y \end{smallmatrix} \right\rangle \circ p_X = y \circ \alpha_X \\ \\ \left\langle \begin{smallmatrix} x \\ y \\ y \end{smallmatrix} \right\rangle \circ p_X = x \circ \alpha_X \end{cases}$$

Universal means $A(X)$ limit of outer square

Theorem [Bourn & Janelidze, 2008]

Let $\mathcal{A}$ be a regular category with binary coproducts. TFAE:

1. If (α, p) is universal, then α is a regular epimorphism;
2. there exists an approximate Mal'tsev co-operation such that $\alpha\colon A \Rightarrow 1_{\mathcal{A}}$ is a regular epimorphism;
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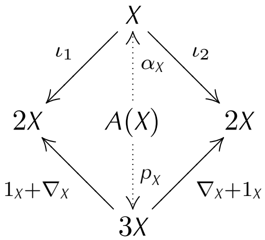

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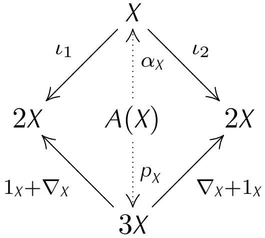

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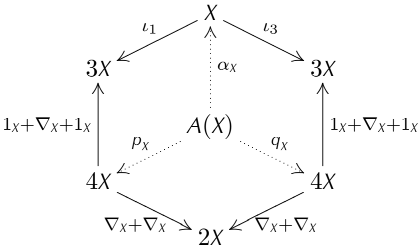
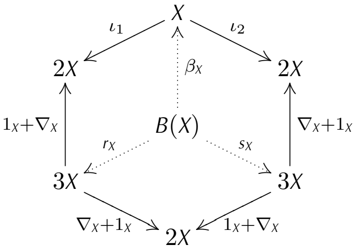


quaternary



ternary

Theorem

Let $\mathcal{A}$ be a regular category with binary coproducts. TFAE:

1. If α or β is universal, then it is a regular epimorphism;
2. there exist approximate Goursat co-operations such that α and β are regular epimorphisms;
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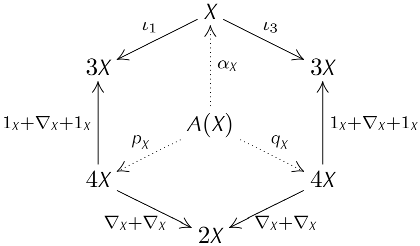

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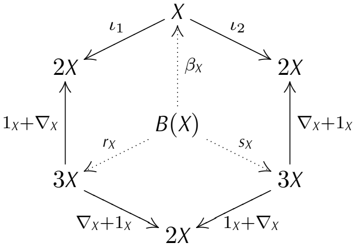

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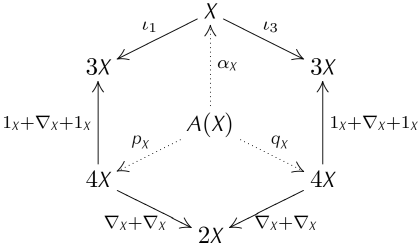

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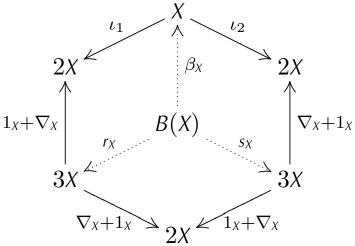

ternary

Theorem

Let $\mathcal{A}$ be a regular category with binary coproducts. TFAE:

- **3** $\mathcal{A}$ is a Goursat category;
- **4** every reflexive relation R satisfies $R^{\mathrm{op}} \leqslant R^2$. □

What about condition 5?

- **5** Every reflexive relation R satisfies $R^3 \leqslant R^2$.

Follows from the characterisation of 4-permutability!

Approximate Goursat co-operations

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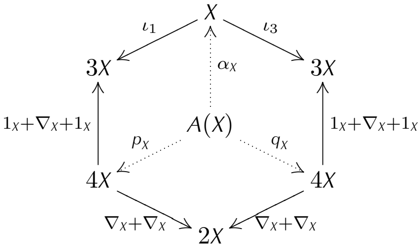

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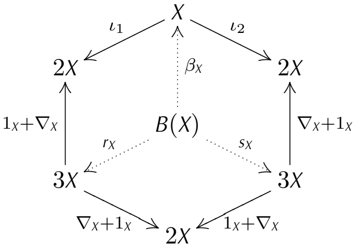

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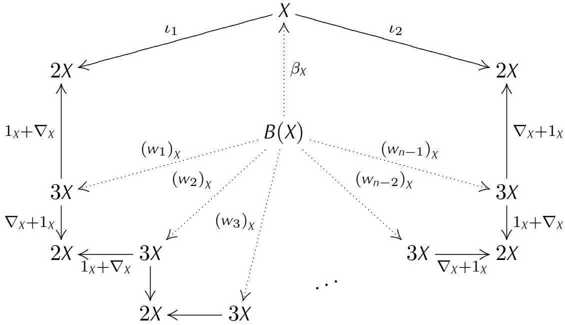
Main theorem: n-permutability

Natural **approximate ternary co-operations** on $\mathcal{A}$, for $n \geqslant 2$:



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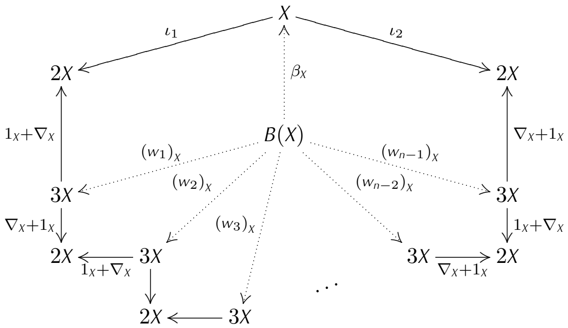

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2. there exist approximate co-operations with α and β regular epi;
3. $\mathcal{A}$ is an n-permutable category. □

Main theorem: n-permutability

Theorem

A regular category with binary coproducts is n-permutable if and only if every reflexive relation R satisfies $R^{\mathrm{op}} \leqslant R^{n-1}$. □

Lemma

If every reflexive relation R in $\mathcal{A}$ satisfies $R^n \leqslant R^{n-1}$ then $\mathcal{A}$ is $(2n-2)$-permutable. □

Theorem

A regular category $\mathcal{A}$ with binary coproducts is n-permutable if and only if every reflexive relation R satisfies $R^n \leqslant R^{n-1}$.

Proof of $\Leftarrow$ in the Goursat case, $n = 3$.

$R^3 \leqslant R^2$ implies that $\mathcal{A}$ is $2 \cdot 3 - 2 = 4$-permutable, so
$R^{\mathrm{op}} \leqslant R^{4-1} = R^3 \leqslant R^2 = R^{3-1}$,
which gives 3-permutability. □

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Conclusion

- Hagemann and Mitschke's theorem has a categorical counterpart:

Theorem [Rodelo & VdL, 2012]

For any regular category with binary sums $\mathcal{A}$ and any $A \in \mathcal{A}$, TFAE:

1. the equivalence relations on A are n-permutable;
2. every reflexive relation R on A satisfies $R^{\mathrm{op}} \leqslant R^{n-1}$;
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- n-permutable categories with finite sums can be characterised in terms of approximate co-operations
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Further questions

- Do we really need binary sums?
  - Counterexamples seem hard to construct:
    - varieties have sums
    - just “taking all finite algebras” or so will not work
  - Embedding theorem for *n*-permutable categories?
- Direct and simple “purely categorical” proof?
  - Closedness properties of relations
- How general is this technique?
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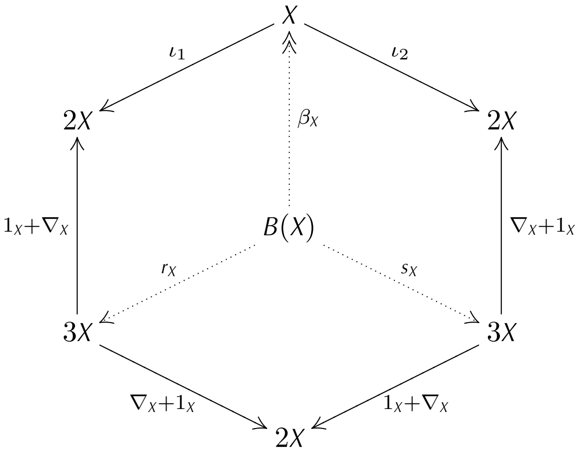

X
ι1
ι2
2X
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1X+∇X
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3X
∇X+1X
1X+∇X
2X