An Indexed Central Limit Theorem

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Standard triangular arrays

Standard triangular array (STA): a triangular array of real square integrable random variables

$$\begin{array}{ccc} \xi_{1,1} & & \\ \xi_{2,1} & \xi_{2,2} & \\ \xi_{3,1} & \xi_{3,2} & \xi_{3,3} \\ & \vdots & \end{array}$$

satisfying the following properties.

$$\begin{array}{ll} (a) & \forall n : \xi_{n,1}, \ldots, \xi_{n,n} \text{ are independent} \\ (b) & \forall n, k : \mathbb{E}\left[\xi_{n,k}\right] = 0 \\ (c) & \forall n : \sum_{k=1}^{n} \sigma_{n,k}^2 = 1, \text{ where } \sigma_{n,k}^2 = \mathbb{E}\left[\xi_{n,k}^2\right] \\ (d) & \max_{k=1}^{n} \sigma_{n,k}^2 \to 0 \end{array}$$

The Lindeberg-Feller CLT

Theorem

Given an STA $(\xi_{n,k})_{n,k}$ *and a normally distributed random variable* ξ*: if*

$$\forall \epsilon > 0 : \sum_{k=1}^{n} \mathbb{E}\left[\xi_{n,k}^2; |\xi_{n,k}| \geq \epsilon\right] \to 0$$

then

$$\sum_{k=1}^{n} \xi_{n,k} \xrightarrow{w} \xi.$$

Lindeberg index

$$\mathrm{Lin}\left(\{\xi_{n,k}\}\right) = \sup_{\epsilon>0} \limsup_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}\left[\xi_{n,k}^2; |\xi_{n,k}| \geq \epsilon\right]$$

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Fix $0 < \alpha < 1$, let $\beta = \frac{\alpha}{1-\alpha}$ and put

$$s_n^2 = (1+\beta)n - \beta \sum_{k=1}^{n} k^{-1} = n + \beta \sum_{k=1}^{n} \left(1 - k^{-1}\right)$$

$$\mathbb{P}\left[\eta_{\alpha,n,k} = -1/s_n\right] = \mathbb{P}\left[\eta_{\alpha,n,k} = 1/s_n\right] = \frac{1}{2}\left(1 - \beta k^{-1}\right)$$

$$\mathbb{P}\left[\eta_{\alpha,n,k} = -\sqrt{k}/s_n\right] = \mathbb{P}\left[\eta_{\alpha,n,k} = \sqrt{k}/s_n\right] = \frac{1}{2}\beta k^{-1}$$

$$\mathrm{Lin}\left(\{\eta_{\alpha,n,k}\}\right) = \alpha$$

Steps in the proof

$$K\left(\eta, \eta'\right) = \sup_{x \in \mathbb{R}} \left|\mathbb{P}[\eta \leq x] - \mathbb{P}[\eta' \leq x]\right|$$

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$\mathcal{H}$: all strictly decreasing functions $h:\mathbb{R}\to\mathbb{R}$, bounded first and second derivatives and a bounded and piecewise continuous third derivative, $\lim_{x\to-\infty}h(x)=1$ and $\lim_{x\to\infty}h(x)=0$.

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$\mathcal{H}$: all strictly decreasing functions $h : \mathbb{R} \to \mathbb{R}$, bounded first and second derivatives and a bounded and piecewise continuous third derivative, $\lim_{x \to -\infty} h(x) = 1$ and $\lim_{x \to \infty} h(x) = 0$.

Step 1

If η is continuously distributed, then the formula

$$\limsup_{n \to \infty} K(\eta, \eta_n) = \sup_{h \in \mathcal{H}} \limsup_{n \to \infty} \left| \mathbb{E}\left[h(\eta) - h(\eta_n) \right] \right|$$

is valid for any sequence $(\eta_n)_n$

Step 2

Let $h : \mathbb{R} \to \mathbb{R}$ *be measurable and bounded. Put*

$$f_h(x) = e^{x^2/2} \int_{-\infty}^{x} (h(t) - \mathbb{E}[h(\xi)])\, e^{-t^2/2} dt$$

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(Basic points of Stein's method)
(1) For any $x \in \mathbb{R}$

$$\mathbb{E}\left[h(\xi)\right] - h(x) = x f_h(x) - f_h'(x).$$

(2) Moreover,

$$\left\| f_h'' \right\|_{\infty} \leq 2 \left\| h' \right\|_{\infty},$$

(3) If $h_z = 1_{]-\infty, z]}$ *for* $z \in \mathbb{R}$, *then for all* $x, y \in \mathbb{R}$

$$\left| f_{h_z}'(x) - f_{h_z}'(y) \right| \leq 1.$$

Step 3

Let $h \in \mathcal{H}$ and put

$$\delta_{n,k} = f_h\left(\sum_{i\neq k}\xi_{n,i} + \xi_{n,k}\right) - f_h\left(\sum_{i\neq k}\xi_{n,i}\right) - \xi_{n,k} f_h'\left(\sum_{i\neq k}\xi_{n,i}\right)$$

$$\epsilon_{n,k} = f_h'\left(\sum_{i\neq k}\xi_{n,i} + \xi_{n,k}\right) - f_h'\left(\sum_{i\neq k}\xi_{n,i}\right) - \xi_{n,k} f_h''\left(\sum_{i\neq k}\xi_{n,i}\right)$$

Then

$$\mathbb{E}\left[\left(\sum_{k=1}^{n}\xi_{n,k}\right) f_h\left(\sum_{k=1}^{n}\xi_{n,k}\right) - f_h'\left(\sum_{k=1}^{n}\xi_{n,k}\right)\right]$$

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Then

$$\mathbb{E}\left[\left(\sum_{k=1}^{n}\xi_{n,k}\right) f_h\left(\sum_{k=1}^{n}\xi_{n,k}\right) - f_h'\left(\sum_{k=1}^{n}\xi_{n,k}\right)\right]$$

$$= \sum_{k=1}^{n}\mathbb{E}\left[\xi_{n,k}\delta_{n,k}\right] - \sum_{k=1}^{n}\sigma_{n,k}^2\mathbb{E}\left[\epsilon_{n,k}\right]$$

Step 4

Let $f : \mathbb{R} \to \mathbb{R}$ *have a bounded derivative and a bounded and piecewise continuous second derivative. Then for any* $a, x \in \mathbb{R}$

$$\left|f(a+x) - f(a) - f'(a)x\right| \\ \leq \min\left\{\left(\sup_{x_1,x_2\in\mathbb{R}}\left|f'(x_1) - f'(x_2)\right|\right)|x|, \frac{1}{2}\left\|f''\right\|_\infty x^2\right\}$$

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Step 5

Let $h \in \mathcal{H}$. Then for all $x, y \in \mathbb{R}$

$$\left|f_h'(x) - f_h'(y)\right| \leq 1$$

Step 6

$$
\begin{aligned}
|\mathbb{E}[h(\xi) - h(\sum_{k=1}^{n} \xi_{n,k})]| &\leq \cdots \\
&\leq \cdots \\
&\leq \frac{1}{2} \left\| f_h'' \right\|_\infty \sum_{k=1}^{n} \mathbb{E}\left[\left|\xi_{n,k}\right|^3 ; |\xi_{n,k}| < \epsilon \right] \\
&+ \left(\sup_{x_1, x_2 \in \mathbb{R}} \left| f_h'(x_1) - f_h'(x_2) \right| \right) \sum_{k=1}^{n} \mathbb{E}\left[\left|\xi_{n,k}\right|^2 ; |\xi_{n,k}| \geq \epsilon \right] \\
&+ \left(\sup_{x_1, x_2 \in \mathbb{R}} \left| f_h''(x_1) - f_h''(x_2) \right| \right) \sum_{k=1}^{n} \sigma_{n,k}^2 \mathbb{E}\left[|\xi_{n,k}| \right] \\
&\leq \cdots
\end{aligned}
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An inequality

Theorem

Given an STA $(\xi_{n,k})_{n,k}$ *and a normally distributed* ξ *the inequality*

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$\mathcal{F}$: probability distributions

$\mathcal{F}_c$: continuous probability distributions

$*$: convolution

Bergström's direct convolution method

$$\eta_n \to \eta \text{ (weak)} \quad \Leftrightarrow \quad \forall \zeta \in \mathcal{F}_c : \eta_n * \zeta \to \eta * \zeta \text{ (uniformly)}$$

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In App $\quad ((\mathcal{F}, \delta_w) \to (\mathcal{F}, K) : \eta \mapsto \eta * \zeta)_{\zeta \in \mathcal{F}_c}$

Distance δ_w

$$\delta_w(\eta, \mathcal{D}) = \sup_{\mathcal{F}_0 \subset \mathcal{F}_c \, finite} \inf_{\psi \in \mathcal{D}} \sup_{\zeta \in \mathcal{F}_0} K(\eta * \zeta, \psi * \zeta)$$

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Limit associated with δ_w

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A sidestep concerning the naturality of δ_w

Tightness: a collection $\mathcal{D}$ of probability distributions is said to be tight if for every $\varepsilon > 0$ there exists a constant $M > 0$ such that for all $\mathcal{F} \in \mathcal{D}$

$$\mathcal{F}(-M) \vee (1 - \mathcal{F}(M)) \leq \varepsilon$$

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$\mathcal{D}$ is relatively compact in the weak topology if and only if it is tight.

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$$\chi_c(A) := \sup_{\mathcal{U} \in \mathsf{U}(A)} \inf_{x \in A} \lambda\mathcal{U}(x)$$

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(Functional approach to topology, M.M. Clementino, E. Giuli, W. Tholen, 2003, Cambridge University Press)

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Indexed Prohorov theorem

In $(\mathcal{F}, \delta_w)$ *for any* $\mathcal{D} \subset \mathcal{F}$*:*

$$\chi_{rc}(\mathcal{D}) = e(\mathcal{D})$$

Back to the CLT

$j(\eta)$: the supremum of the discontinuity jumps of the distribution of η

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$$\lambda_w(\eta_n)(\eta) = \sup_{\zeta \in \mathcal{F}_c} \limsup_{n\to\infty} K(\eta * \zeta, \eta_n * \zeta)$$

Step 7

For $\eta \in \mathcal{F}$ and $(\eta_n)_n$ in $\mathcal{F}$

$$\lambda_w(\eta_n)(\eta) \leq \limsup_{n\to\infty} K(\eta, \eta_n) \leq \lambda_w(\eta_n)(\eta) + j(\eta)$$

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Consequence

For $\eta \in \mathcal{F}_c$ and $(\eta_n)_n$ in $\mathcal{F}$

$$\lambda_w(\eta_n)(\eta) = \limsup_{n \to \infty} K(\eta, \eta_n)$$

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Given an STA $(\xi_{n,k})_{n,k}$ and a normally distributed ξ the inequality

$$\lambda_w\left(\sum_{k=1}^{n} \xi_{n,k}\right)(\xi) \leq \mathrm{Lin}\left(\{\xi_{n,k}\}\right)$$

is valid.

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