.

Notes on exact meets and joins

R. N. Ball, J. Picado and A. Pultr

Exact meets and joins.

Recall the following operations

$$
\begin{aligned}
a \downarrow b &= \{x \mid x \wedge a \leq b\} \quad \text{and} \\
a \uparrow b &= \{x \mid x \vee a \geq b\}.
\end{aligned}
$$

An element b is the *exact meet* of a subset A of a lattice L if

- b is a lower bound of A, and
- for any $c < d$, if $A \subseteq c \uparrow d$ then $b \in c \uparrow d$.

(the latter in detail: if $a \vee c \geq d$ for all $a \in A$ then $b \vee c \geq d$).

Dually, b is the *exact join* of a subset A if

- b is an upper bound of A, and
- for any $c < d$, if $A \subseteq c \downarrow d$ then $b \in c \downarrow d$.

(Bruns and Lakser speak of *admissible* joins.)

Open and closed sublocales. G-sets.

In the co-frame $\mathcal{S}\ell(L)$ of all sublocales of a locale L we have in particular

– the open sublocale associated with the elements $a \in L$

$$\mathfrak{o}(a) = \{a \to x \mid x \in L\} = \{x \mid a \to x = x\}.$$

– and their complements, the *closed* sublocales

$$\mathfrak{c}(a) = \uparrow a.$$

In the co-frame $\mathcal{S}\ell(L)$ we have

$$\bigvee_{i \in J} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i \in J} a_i), \quad \mathfrak{o}(a) \wedge \mathfrak{o}(b) = \mathfrak{o}(a \wedge b),$$

$$\bigwedge_{i \in J} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i \in J} a_i), \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b).$$

For purposes of the discussion of more general (bounded) lattices L we will generalize the sublocales to the *geometric subsets* (briefly, *G-subsets*) the subsets $S \subseteq L$ such that

$$\text{if } M \subseteq S \text{ and } \bigwedge M \text{ exists then } \bigwedge M \in S.$$

The system of all G-subsets of a lattice L will be denoted by

$$\mathcal{G}(L)$$

and following the situation from $\mathcal{S}\ell(L)$ of a frame we speak of the subsets

$$\mathfrak{c}(a) = \uparrow a$$

as of the *closed* G-subsets.

Proposition. *For any lattice, $\mathcal{G}(L)$ ordered by inclusion is a complete lattice with the join*

$$\bigvee_{i\in J} S_i = \{\bigwedge M \mid M \subseteq \bigcup_{i\in J} S_i,\ \bigwedge M \text{ exists}\}.$$

Consequently, if L is a frame, the sublocale coframe $\mathcal{S}\ell(L)$ is a subset of $\mathcal{G}(L)$ closed under all joins.

Proposition. *Let $\bigvee_{i\in J}\mathfrak{c}(a_i)$ in $\mathcal{G}(L)$ be closed. Then $a = \bigwedge_i a_i$ exists, and $\bigvee_{i\in J}\mathfrak{c}(a_i) = \mathfrak{c}(a)$.*

Theorem. *A meet $\bigwedge A$ in L is exact if and only if the join $S = \bigvee\{\mathfrak{c}(a) \mid a \in A\}$ is closed.*

Exact meets in frames

Fact. *In the co-frame* $\mathcal{S}\ell(L)$ *we have for the pseudosupplement* $x^{\#}$, $\bigvee \mathfrak{c}(a_i) = \mathfrak{c}(a)$ *if and only if* $(\bigcap \mathfrak{o}(a_i))^{\#\#} = \mathfrak{o}(a)$.

Theorem. *TFAE:*

(1) *The meet* $a = \bigwedge_i a_i$ *is exact.*

(2) *For all* $b \in L$, $(\bigwedge_i a_i) \vee b = \bigwedge_i (a_i \vee b)$.

(3) $\bigwedge a_i = a$ *and* $\bigvee_i \mathfrak{c}(a_i) = \mathfrak{c}(a)$ *in* $\mathcal{S}\ell(L)$.

(4) $\bigwedge a_i = a$ *and* $\bigvee_i \mathfrak{c}(a_i)$ *is closed.*

(5) *If* $x \geq \bigwedge_i a_i$ *then there exist* $x_i \geq a_i$ *such that* $x = \bigwedge_i x_i$.

(6) $(\bigwedge_i \mathfrak{o}(a_i))^{\#\#} = (\bigcap_i \mathfrak{o}(a_i))^{\#\#} = \mathfrak{o}(a)$ *in* $\mathcal{S}\ell(L)$.

(7) $(\bigwedge_i \mathfrak{o}(a_i))^{\#\#} = (\bigcap_i \mathfrak{o}(a_i))^{\#\#}$ *is an open sublocale of* L.

Strongly exact (free) meets.

Instead of closed joins of closed sublocales, we require the meets (intersections) of open sublocales to be open:

$$\bigwedge_{i\in J} \mathfrak{o}(a_i) = \bigcap_{i\in J} \mathfrak{o}(a_i) = \mathfrak{o}(a). \qquad \text{(s-exact)}$$

Theorem. *The following facts about a meet $a = \bigwedge_{i\in J} a_i$ in a frame L are equivalent.*

(1) *The meet $a = \bigwedge_i a_i$ is strongly exact.*

(2) $\bigwedge_i \mathfrak{o}(a_i) = \bigcap_i \mathfrak{o}(a_i)$ *is open.*

(3) *If $a_i \to x = x$ for all $i \in J$ then*

$$(\bigwedge_{i\in J} a_i) \to x = x.$$

Strongly exact meets appeared, viewed from another perspective, as the

free meets

in the unpublished Thesis by Todd Wilson: *the meets that are preserved by all frame homomorphisms.*

Wilson proved, a.o., the equivalent of (s-exact) as one of the characteristics of the freeness. Wilson's characteristic, slightly modified:

Theorem. *TFAE:*

(1) $\bigwedge_i a_i$ *is strongly exact.*

(2) *for every frame homomorphism* $h\colon L \to M$, $h(\bigwedge_i a_i) = \bigwedge_i h(a_i)$ *and it is strongly exact.*

(3) *for every frame homomorphism* $h\colon L \to M$, $h(\bigwedge_i a_i) = \bigwedge_i h(a_i)$.

(4) *For every* $x \in L$, $\bigwedge_{a\in A}(a \to x) \to x = ((\bigwedge A) \to x) \to x$.

Note. $\mathcal{N}(L) = \mathcal{S}\ell(L)^{\mathrm{op}}$ is a frame and we have a frame homomorphism

$$c_L = (a \mapsto \mathfrak{o}(a)) \colon L \to \mathcal{N}(L)$$

starting the famous Assembly Tower. Another characteristic of the strongly exact (free) meets Todd Wilson presented was that

$$\bigwedge \mathfrak{c}_{\mathcal{N}(L)} c_L[A] = \mathfrak{c}_{\mathcal{N}(L)} c_L(\bigwedge A)$$

in $\mathcal{N}^2(L)$.

Conservative subsets.

(1) In our language, a subset $A \subseteq L$ is *conservative* iff B is exact for all $B \subseteq A$. Dowker and Papert (1975) and Chen (1992) used conservative subsets of frames in the study of paracompactness.

(2) Exact meets are also related with the concepts of *interior-preserving* and *closure-preserving* families of sublocales of Plewe. A family

$$\mathcal{S} = \{S_i \mid i \in I\} \subseteq \mathcal{S}\ell(L)$$

is *closure-preserving* if for all $J \subseteq I$,

$$\mathsf{cl}(\bigvee_{i\in J} S_i) = \bigvee_{i\in J} \mathsf{cl}(S_i).$$

Dually, $\mathcal{S}$ is *interior-preserving* if for all $J \subseteq I$,

$$\mathsf{int}(\bigwedge_{i\in J} S_i) = \bigwedge_{i\in J} \mathsf{int}(S_i).$$

Then, a subset A of L is said to be *interior-preserving* (resp. *closure-preserving*) if

$$\{\mathfrak{o}(a) \mid a \in A\} \text{ is interior-preserving}$$

$$(\text{resp. } \{\mathfrak{c}(a) \mid a \in A\} \text{ is closure-preserving}).$$

Interior-preserving covers play a decisive role in the construction of (canonical) examples of transitive quasi-uniformities for frames (Ferreira and Picado).

Any interior-preserving cover of L is closure-preserving but somewhat surprising, contrarily to what happens in the classical case, the converse does not hold in general (the cover $\mathbb{N}$ of the frame $L = (\omega + 1)^{op} = \{\infty < \cdots < 2 < 1\}$ is such an example).

Lemma. *Let $A \subseteq L$. Then:*

(1) *A is interior-preserving iff*

$$\bigwedge_{b\in B} \mathfrak{o}(b) = \mathfrak{o}(\bigwedge B)$$

for every $B \subseteq A$.

(2) *A is closure-preserving iff*

$$\bigvee_{b\in B} \mathfrak{c}(b) = \mathfrak{c}(\bigwedge B)$$

for every $B \subseteq A$.

Corollary. *A subset A of a frame L is conservative if and only if it is closure-preserving.*

This gives an example of an $A \subseteq L$ for which any $B \subseteq A$ is exact but, being not interior-preserving, such that there is some $B \subseteq A$ which is not strong exact. Thus

> *strong exactness is indeed a stronger property than exactness.*

Exact and strongly exact in spaces.

A space X is T_D if

$\forall x \in X\ \exists U \ni x$ open such that $U \smallsetminus \{x\}$ is open.

(Aull and Thron 1963, also Bruns 1962.)

Proposition. *A space is T_D iff there holds the equivalence*

$$(\forall A \text{ open},\ \mathsf{int}\,(U \cup A) = \mathsf{int}\, U \cup A\ ) \quad \text{iff} \quad U \text{ is open.}$$

Corollary. *A space is T_{D-0} iff for every $\sim$-set U there holds the equivalence*

$$(\forall A \text{ open},\ \mathsf{int}\,(U \cup A) = \mathsf{int}\, U \cup A\ ) \quad \text{iff} \quad U \text{ is open.}$$

Lemma. *In any space X,*

$$\mathsf{int}\, U = \bigwedge \{X \smallsetminus \overline{\{x\}} \mid x \notin U\}.$$

Theorem. *TFAE for a topological space X.*

(1) *X is T_{D-0}.*

(2) *A meet $\bigwedge U_i$ is exact in $\Omega(X)$ iff $\bigcap U_i$ is open.*

What this says about intersections of open subsets in spaces: A subset A of a topological space X induces a congruence

$$E_A = \{(U, V) \mid U \cap A = V \cap A\}.$$

Write

$$A \sim B \quad \text{for} \quad E_A = E_B.$$

For T_D-spaces we have $A \sim B$ iff $A = B$, and this fact holds only in T_D-spaces. In fact, one needs T_D even for the special case when one of the A, B is open. Thus the facts above can be directly interpreted in spaces only as the statement that

> *if X is a T_D-space then the meet $\bigwedge U_i$ in $\Omega(X)$ is strongly exact iff $\bigwedge \mathfrak{o}(U_i)$ is open.*

BUT :

Lemma. *Let X be an arbitrary space, A, W subsets, $A \sim W$ and W open. Then for each open subset $U \subseteq X$ we have*

$$A \subseteq U \quad \textit{iff} \quad W \subseteq U.$$

Corollary. *For any topological space X the meet $\bigwedge U_i$ in $\Omega(X)$ is strongly exact iff $\bigcap U_i$ is open.*

Corollary. *If X is not T_{D-0} then the exactness and strong exactness in $\Omega(X)$ differ.*

Theorem. *A spatial L is T_D-spatial iff strongly exact meets and exact meets in L coincide.*

Note: Exact meets of meet irreducible elements.

View ΣL as the set of all meet-irreducible elements $p \in L$ and take the natural isomorphism for L spatial,

$$(a \mapsto \Sigma_a)\colon L \cong \Omega\Sigma L.$$

We obtain that

> *$a = \bigwedge_i a_i$ is strongly exact iff $\Sigma_a = \bigwedge_i \Sigma_{a_i}$ is strongly exact.*

Hence

> *$a = \bigwedge_i a_i$ is strongly exact iff $\Sigma_a = \bigcap_i \Sigma_{a_i}$.*

Thus, $\bigwedge_i a_i \leq p$ iff $\exists j,\ a_j \leq p$, which reduces to the implication

$$\bigwedge_i a_i \leq p \quad \Rightarrow \quad \exists j,\ a_j \leq p.$$

For an $L \cong \Omega(X)$ with a T_D-space X, this is then another criterion of exactness.

Strongly exact in Scott topology

The set of all up-sets will be denoted by

$$\mathfrak{U}(X).$$

The *Scott topology* σ_X on a lattice X consists of the $U \in \mathfrak{U}(X)$ such that

$$\bigvee D \in U \;\Rightarrow\; D \cap U \neq \emptyset$$

for any directed $D \subseteq X$.

Now the spectrum of a frame L will be represented as the set $\Sigma' L$ of all completely prime filters P in L endowed with the topology consisting of the open sets $\Sigma'_a = \{P \mid a \in P\}$, $a \in L$. Each $P \in \Sigma' L$ is Scott open in L.

More generally, in a lattice L we will consider the pre-topology

$$\Sigma'_L = \{\Sigma'_x \,|\, x \in L\}, \quad \Sigma'_x = \{U \in \mathfrak{U}(L) \,|\, x \in U\}.$$

One of the important facts needed in the proof of the Hofmann-Lawson duality is that

> *an intersection* $\bigcap \mathcal{P}$ *of a set of completely prime filters is Scott open* (*that is,* $\bigwedge \mathcal{P}$ *is strongly exact in* σ_L) *iff* $\mathcal{P}$ *is a compact subset of* $\Sigma' L$.

This is a part of a more general fact.

A subset $\mathcal{U}$ of $\mathfrak{U}(L)$ is *d-compact* if one can choose in every directed cover of $\mathcal{U}$ by the element of Σ'_L an element covering $\mathcal{U}$.

Proposition. *Let a set $\mathcal{U}$ of Scott open sets be d-compact in Σ'_L. Then $\bigcap\mathcal{U}$ is Scott open, and hence $\bigwedge\mathcal{U}$ is strongly exact in σ_L.*

Proposition. *Let $X = (X, \leq)$ be a complete lattice. Let $\mathcal{U}$ be a set of Scott open sets in X and let $\bigcap\mathcal{U}$ be Scott open. Then $\mathcal{U}$ is d-compact in Σ'_X.*

Proposition. *Let L be a complete lattice. Then a meet $\bigwedge\mathcal{U}$ in the Scott topology σ_L is strongly exact iff $\mathcal{U}$ is d-compact in the pre-topology Σ'_L on $\mathfrak{U}(L)$.*

More about exactness and maps

Each frame homomorphism preserves all strongly exact meets (Wilson) and this characterizes strong exactness. Consequently, no such universal behaviour can be expected from the plain exactness.

First, however, we apply this to T_D-spaces. We obtain

Corollary. *Let L be T_D-spatial. Then each frame homomorphism $h\colon L \to M$ sends all exact meets in L to strongly exact meets in M.*

Co-weakly open homomorphisms.

A frame homomorphism h is *weakly open* if

$$h(x^{**}) \leq h(x)^{**}.$$

h is *co-weakly open* if for the associated coframe homomorphism f_{-1} and the pseudosupplement $S^{\#}$ one has

$$f_{-1}(S)^{\#\#} \subseteq f_{-1}(S^{\#\#}).$$

Proposition. *A co-weakly open homomorphism preserves all exact meets.*

Closed localic maps.

A localic map $f\colon L \to M$ is *closed* if the image of each closed sublocale is closed; that is,

$$f[\mathfrak{c}(a)] = \mathfrak{c}(f(a)) \text{ for each } a \in L.$$

Or: f is closed if and only if for its left adjoint h,

$$c \le f(a) \vee b \quad \text{iff}$$
$$\forall a \in L\ \forall b, c \in M,\ h(c) \le a \vee h(b).$$

Proposition. *A closed localic map preserves all exact meets.*

A consequence of this fact is the extension to frames of the result of Michael (1967) that the image of a paracompact space under a continuous closed map is paracompact.

Recall that a subset U of L is a *closed covering* if

$$x = \bigwedge_{u \in U} (x \vee u) \text{ for every } x \in L.$$

A closed covering is a *dual-refinement* of a cover A if for each $u \in U$ there exists $a \in A$ such that $u \vee a = 1$.

By a result of Dowker and Papert a frame L is paracompact and normal iff each cover A of L has a conservative dual-refinement.

Corollary. *The image of a normal paracompact frame under a closed localic mapping is paracompact.*