Projections of weak braided Hopf algebras

Carlos Soneira Calvo

Univ. da Coruña

Workshop on Category Theory. Universidade de Coimbra

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Motivation

Weak Hopf algebras

- G. Böhm, F. Nill, K. Szlachányi, *Weak Hopf algebras I. Integral theory and C^*-structure*, J. of Algebra **221** (1999)

Applications:

- Operators theory
- Extension of algebras
- Space-time models in dimension 2
- Solutions to the dynamic Yang-Baxter equation
- Reconstruction of categories

Hypothesis

Hypothesis

We consider a monoidal category $\mathcal{C}$:

- ***strict***
- ***with split idempotents:***
  For any $\nabla : A \to A$ with $\nabla = \nabla \circ \nabla$, exist
  $B \in |\mathcal{C}|, i : B \to A, p : A \to B$ such that $\nabla = i \circ p$ y $p \circ i = id_B$

Algebra in $\mathcal{C}$: a triple $A = (A, \eta_A, \mu_A)$ with $A \in |\mathcal{C}|$ and $\eta_A \in \mathcal{C}(A, K)$ (**unit**), $\mu_A \in \mathcal{C}(D \otimes D, D)$ (**product**) such that

$\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$ and $\mu_A \circ (\mu_A \otimes A) = \mu_A \circ (A \otimes \mu_A)$.

Preliminaries

Algebra in $\mathcal{C}$: a triple $A = (A, \eta_A, \mu_A)$ with $A \in |\mathcal{C}|$ and $\eta_A \in \mathcal{C}(A, K)$ (unit), $\mu_A \in \mathcal{C}(A \otimes A, A)$ (product) such that

$$\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A) \text{ y } \mu_A \circ (\mu_A \otimes A) = \mu_A \circ (A \otimes \mu_A).$$

If A, B algebras, $f \in \mathcal{C}(A, B)$ is a morphism of algebras if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $\eta_B = f \circ \eta_A$.

Coalgebra in $\mathcal{C}$: a triple $D = (D, \varepsilon_D, \delta_D)$ with $D \in |\mathcal{C}|$ and $\varepsilon_D \in \mathcal{C}(D, K)$ (counit), $\delta_D \in \mathcal{C}(D, D \otimes D)$ (coproduct) such that

$$(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D \text{ and } (\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D.$$

If D, E are coalgebras, $f \in \mathcal{C}(D, E)$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$.

If A algebra, B coalgebra and $f, g \in \mathcal{C}(B, A)$, the convolution of f and g is $f \wedge g = \mu_A \circ (f \otimes g) \circ \delta_B$.

Preliminaries

Let A be an algebra. A left A-module is a pair (M, φ_M) with:

$$M \in |\mathcal{C}|, \quad \varphi_M \in \mathcal{C}(A \otimes M, M) \text{ such that}$$

$$\varphi_M \circ (\eta_A \otimes M) = id_M, \quad \varphi_M \circ (A \otimes \varphi_M) = \varphi_M \circ (\mu_A \otimes M).$$

Given (M, φ_M), (N, φ_N), left A-mod, $f : M \to N$ is an A-mod morphism if

$$\varphi_N \circ (A \otimes f) = f \circ \varphi_M.$$

Let D be a coalgebra. A left D-comodule is a pair (M, ϱ_M) with:

$$M \in |\mathcal{C}|, \quad \varrho_M \in \mathcal{C}(M, D \otimes M) \text{ such that}$$

$$(\varepsilon_D \otimes M) \circ \varrho_M = id_M, \quad (D \otimes \varrho_M) \circ \varrho_M = (\delta_D \otimes M) \circ \varrho_M.$$

Given (M, ϱ_M), (N, ϱ_N), left D-comod, $f : M \to N$ is a morphism of D-comod if

$$\varrho_N \circ f = (D \otimes f) \circ \varrho_M.$$

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Weak braided Hopf algebras

Definition

Let $D \in |\mathcal{C}|$ and $t_{D,D} : D \otimes D \to D \otimes D$ morfismo en $\mathcal{C}$. It is said that $t_{D,D}$ satisfies the Yang-Baxter equation if

$$(t_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) = (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}).$$

Definition (Weak Yang-Baxtder operator)

Let $D \in |\mathcal{C}|$. A weak Yang-Baxter operator is a morphism $t_{D,D} : D \otimes D \to D \otimes D$ in $\mathcal{C}$ such that:

(a1) $t_{D,D}$ satisfies the Yang-Baxter equation.

(a2) There exists an idempotent $\nabla_{D,D} : D \otimes D \to D \otimes D$ such that:

(a2-1) $(\nabla_{D,D} \otimes D) \circ (D \otimes \nabla_{D,D}) = (D \otimes \nabla_{D,D}) \circ (\nabla_{D,D} \otimes D)$,

(a2-2) $(\nabla_{D,D} \otimes D) \circ (D \otimes t_{D,D}) = (D \otimes t_{D,D}) \circ (\nabla_{D,D} \otimes D)$,

(a2-3) $(t_{D,D} \otimes D) \circ (D \otimes \nabla_{D,D}) = (D \otimes \nabla_{D,D}) \circ (t_{D,D} \otimes D)$,

(a2-4) $t_{D,D} \circ \nabla_{D,D} = \nabla_{D,D} \circ t_{D,D} = t_{D,D}$.

Weak braided Hopf algebras

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Weak braided Hopf algebras

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- (a2-4) $t_{D,D} \circ \nabla_{D,D} = \nabla_{D,D} \circ t_{D,D} = t_{D,D}$.

Weak braided Hopf algebras

(a3) There exists $t'_{D,D} : D \otimes D \to D \otimes D$ such that:

(a3-1) The morphism $p_{D,D} \circ t_{D,D} \circ i_{D,D} : D \times D \to D \times D$ is an isomorphism with inverse $p_{D,D} \circ t'_{D,D} \circ i_{D,D} : D \times D \to D \times D$, where $p_{D,D}$ e $i_{D,D}$ are those such that $i_{D,D} \circ p_{D,D} = \nabla_{D,D}$ y $p_{D,D} \circ i_{D,D} = id_{D \times D}$ with $D \times D$ the image of $\nabla_{D,D}$.

(a3-2) $t'_{D,D} \circ \nabla_{D,D} = \nabla_{D,D} \circ t'_{D,D} = t'_{D,D}$.

Example

If $(\mathcal{C}, \otimes, K, c_{-,-})$ is a braided category:

- The braid $c_{D,D}$ satisfies the Yang-Baxter equation
- The braid $t_{D,D} := c_{D,D}$ is a weak Yang-Baxter operator with $\nabla_{D,D} = id_{D \otimes D}$, $t'_{D,D} = c^{-1}_{D,D}$

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Weak braided Hopf algebras

Definition (Weak braided bialgebra (WBB))

Is a $D \in \mathcal{C}$ with

algebra-coalgebra strutcture $(D, \eta_D, \mu_D), (D, \varepsilon_D, \delta_D)$

and a weak Yang-Baxter op. $t_{D,D} : D \otimes D \to D \otimes D$ with associated idempotent $\nabla_{D,D}$ such that:

(b1) It holds that

(b1-1) $\mu_D \circ \nabla_{D,D} = \mu_D$,
(b1-2) $\nabla_{D,D} \circ (\mu_D \otimes D) = (\mu_D \otimes D) \circ (D \otimes \nabla_{D,D})$,
(b1-3) $\nabla_{D,D} \circ (D \otimes \mu_D) = (D \otimes \mu_D) \circ (\nabla_{D,D} \otimes D)$.

(b2) It holds that

(b2-1) $\nabla_{D,D} \circ \delta_D = \delta_D$,
(b2-2) $(\delta_D \otimes D) \circ \nabla_{D,D} = (D \otimes \nabla_{D,D}) \circ (\delta_D \otimes D)$,
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(b1-3) $\nabla_{D,D} \circ (D \otimes \mu_D) = (D \otimes \mu_D) \circ (\nabla_{D,D} \otimes D).$

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(b2-1) $\nabla_{D,D} \circ \delta_D = \delta_D,$
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- (b1-2) $\nabla_{D,D} \circ (\mu_D \otimes D) = (\mu_D \otimes D) \circ (D \otimes \nabla_{D,D})$,
- (b1-3) $\nabla_{D,D} \circ (D \otimes \mu_D) = (D \otimes \mu_D) \circ (\nabla_{D,D} \otimes D)$.

(b2) It holds that

- (b2-1) $\nabla_{D,D} \circ \delta_D = \delta_D$,
- (b2-2) $(\delta_D \otimes D) \circ \nabla_{D,D} = (D \otimes \nabla_{D,D}) \circ (\delta_D \otimes D)$,
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Weak braided Hopf algebras

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(b1) It holds that

(b1-1) $\mu_D \circ \nabla_{D,D} = \mu_D,$

(b1-2) $\nabla_{D,D} \circ (\mu_D \otimes D) = (\mu_D \otimes D) \circ (D \otimes \nabla_{D,D}),$

(b1-3) $\nabla_{D,D} \circ (D \otimes \mu_D) = (D \otimes \mu_D) \circ (\nabla_{D,D} \otimes D).$

(b2) It holds that

(b2-1) $\nabla_{D,D} \circ \delta_D = \delta_D,$

(b2-2) $(\delta_D \otimes D) \circ \nabla_{D,D} = (D \otimes \nabla_{D,D}) \circ (\delta_D \otimes D),$

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Is a $D \in \mathcal{C}$ with

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(b1) It holds that

- (b1-1) $\mu_D \circ \nabla_{D,D} = \mu_D$,
- (b1-2) $\nabla_{D,D} \circ (\mu_D \otimes D) = (\mu_D \otimes D) \circ (D \otimes \nabla_{D,D})$,
- (b1-3) $\nabla_{D,D} \circ (D \otimes \mu_D) = (D \otimes \mu_D) \circ (\nabla_{D,D} \otimes D)$.

(b2) It holds that

- (b2-1) $\nabla_{D,D} \circ \delta_D = \delta_D$,
- (b2-2) $(\delta_D \otimes D) \circ \nabla_{D,D} = (D \otimes \nabla_{D,D}) \circ (\delta_D \otimes D)$,
- (b2-3) $(D \otimes \delta_D) \circ \nabla_{D,D} = (\nabla_{D,D} \otimes D) \circ (D \otimes \delta_D)$.

Weak braided Hopf algebras

(b3) It holds that

(b3-1) $t_{D,D} \circ (\mu_D \otimes D) = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D})$,
(b3-2) $t_{D,D} \circ (D \otimes \mu_D) = (\mu_D \otimes D) \circ (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D)$,
(b3-3) $(\delta_D \otimes D) \circ t_{D,D} = (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes \delta_D)$,
(b3-4) $(D \otimes \delta_D) \circ t_{D,D} = (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D)$.

(b4) $\delta_D \circ \mu_D = (\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)$.

(b5) $\varepsilon_D \circ \mu_D \circ (\mu_D \otimes D) = ((\varepsilon_D \circ \mu_D) \otimes (\varepsilon_D \circ \mu_D)) \circ (D \otimes \delta_D \otimes D)$
$= ((\varepsilon_D \circ \mu_D) \otimes (\varepsilon_D \circ \mu_D)) \circ (D \otimes (t'_{D,D} \circ \delta_D) \otimes D)$.

(b6) $(\delta_D \otimes D) \circ \delta_D \circ \eta_D = (D \otimes \mu_D \otimes D) \circ ((\delta_D \circ \eta_D) \otimes (\delta_D \circ \eta_D))$
$= (D \otimes (\mu_D \circ t'_{D,D}) \otimes D) \circ ((\delta_D \circ \eta_D) \otimes (\delta_D \circ \eta_D))$.

(b3) It holds that

(b3-1) $t_{D,D} \circ (\mu_D \otimes D) = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D})$,
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(b3-3) $(\delta_D \otimes D) \circ t_{D,D} = (D \otimes t_{D,D}) \circ (t_{D,D} \otimes D) \circ (D \otimes \delta_D)$,
(b3-4) $(D \otimes \delta_D) \circ t_{D,D} = (t_{D,D} \otimes D) \circ (D \otimes t_{D,D}) \circ (\delta_D \otimes D)$.

(b4) $\delta_D \circ \mu_D = (\mu_D \otimes \mu_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\delta_D \otimes \delta_D)$.

(b5) $\varepsilon_D \circ \mu_D \circ (\mu_D \otimes D) = ((\varepsilon_D \circ \mu_D) \otimes (\varepsilon_D \circ \mu_D)) \circ (D \otimes \delta_D \otimes D)$
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Weak braided Hopf algebras

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(b3-1) $t_{D,D} \circ (\mu_D \otimes D) = (D \otimes \mu_D) \circ (t_{D,D} \otimes D) \circ (D \otimes t_{D,D})$,
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$= (D \otimes (\mu_D \circ t'_{D,D}) \otimes D) \circ ((\delta_D \circ \eta_D) \otimes (\delta_D \circ \eta_D))$.

Definition (Weak braided Hopf algebra (WBHA))

Is a WBB such that in addition it verifies that:

(b7) There exists a morphism $\lambda_D : D \to D$ en $\mathcal{C}$ (the antipode of D) such that:

(b7-1) $id_D \wedge \lambda_D = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes t_{D,D}) \circ ((\delta_D \circ \eta_D) \otimes D)$,
(b7-2) $\lambda_D \wedge id_D = (D \otimes (\varepsilon_D \circ \mu_D)) \circ (t_{D,D} \otimes D) \circ (D \otimes (\delta_D \circ \eta_D))$,
(b7-3) $\lambda_D \wedge id_D \wedge \lambda_D = \lambda_D$.

Weak braided Hopf algebras

If D is a WBB or a WBHA we define the idempotents:

$$\Pi_D^L = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes t_{D,D}) \circ ((\delta_D \circ \eta_D) \otimes D),$$

$$\Pi_D^R = (D \otimes (\varepsilon_D \circ \mu_D)) \circ (t_{D,D} \otimes D) \circ ((D \otimes (\delta_D \circ \eta_D)).$$

and the idempotents $\overline{\Pi}_D^L$ y $\overline{\Pi}_D^R$:

$$\overline{\Pi}_D^L = (D \otimes (\varepsilon_D \circ \mu_D)) \circ ((\delta_D \circ \eta_D) \otimes D),$$

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Weak braided Hopf algebras

If D is a WBB or a WBHA we define the idempotents:

$$\Pi_D^L = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes t_{D,D}) \circ ((\delta_D \circ \eta_D) \otimes D),$$

$$\Pi_D^R = (D \otimes (\varepsilon_D \circ \mu_D)) \circ (t_{D,D} \otimes D) \circ ((D \otimes (\delta_D \circ \eta_D)).$$

and the idempotents $\overline{\Pi}_D^L$ y $\overline{\Pi}_D^R$:

$$\overline{\Pi}_D^L = (D \otimes (\varepsilon_D \circ \mu_D)) \circ ((\delta_D \circ \eta_D) \otimes D),$$

$$\overline{\Pi}_D^R = ((\varepsilon_D \circ \mu_D) \otimes D) \circ (D \otimes (\delta_D \circ \eta_D)).$$

Weak braided Hopf algebras: Examples

Examples

- Classic Hopf algebras, Braided Hopf algebras
- Weak Hopf algebras
- Weak Hopf algebras (WHA for short) in braided categories
  - A WBHA H with:
    weak Yang-Baxter operator $t_{H,H} := c_{H,H}$
    $\nabla_{H,H} = id_{H,H}$
    $t'_{H,H} = c^{-1}_{H,H}$

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Definition (Yetter-Drinfeld modules over WBHA)

Let D be a WBHA. We denote by ${}^{D}_{D}\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over D.

Its objects are the triples (M, φ_M, ρ_M) with (M, φ_M) a left D-mod, (M, ρ_M) a left D-comod and:

(1) $\rho_M = (\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \rho_M) \circ (\eta_D \otimes M)$.

(2) $\exists\ t_{D,M} : D \otimes M \to M \otimes D$ and $t_{M,D} : M \otimes D \to D \otimes M$ such that

$$(\mu_D \otimes M) \circ (D \otimes t_{M,D}) \circ ((\rho_M \circ \varphi_M) \otimes D) \circ (D \otimes t_{D,M}) \circ (\delta_D \otimes M)$$
$$= (\mu_D \otimes \varphi_M) \circ (D \otimes t_{D,D} \otimes M) \circ (\delta_D \otimes \rho_M).$$

Given $M, N \in^{D}_{D} \mathcal{Y}D$, $f : M \to N$ in $\mathcal{C}$ is in ${}^{D}_{D}\mathcal{Y}D(M, N)$ if $f \circ \varphi_M = \varphi_N \circ (D \otimes f)$ and $(D \otimes f) \circ \rho_M = \rho_N \circ f$.

Notation

Notation

Given $M, N \in {}^{D}_{D}\mathcal{YD}$, we consider the idempotent $\nabla_{M\otimes N} : M \otimes N \to M \otimes N$

$$\nabla_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (D \otimes t_{D,M} \otimes N) \circ ((\delta_D \circ \eta_D) \otimes M \otimes N)$$

and denote by $M \times N$, $i_{M\otimes N}$ and $p_{M\otimes N}$ the image and splitting morphisms. Prod

We denote by $\nabla_{M\otimes H}$ the idempotent

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Índice

Projections and WBHA structures

Definition (Projection)

Let D be a WBHA.

A projection over D is a triple (B, f, g) with

$$B \text{ un WBHA},$$

$$f : D \to B, g : B \to D \text{ WBHA morphisms such that } g \circ f = id_D \text{ and:}$$

(i) $(B \otimes (f \circ g)) \circ t_{B,B} = t_{B,B} \circ ((f \circ g) \otimes B)$,

(ii) $((f \circ g) \otimes B) \circ t_{B,B} = t_{B,B} \circ (B \otimes (f \circ g))$.

A morphism $h : B \to B'$ of projections (B, f, g), (B', f', g') is

$$\text{WBHA morphism such that } h \circ f = f', \quad g' \circ h = g.$$

The class of projections and its morphisms constitute $\mathcal{P}roj(D)$.

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Proposition

Let D be a WBHA and $(B, f, g) \in \mathcal{P}roj(D)$.

The morphism $q_D^B := id_B \wedge (f \circ \lambda_D \circ g) : B \longrightarrow B$ is idempotent.

Notation

B_D the image of q_D^B,

$p_D^B : B \to B_D$, $\quad i_D^B : B_D \to B$ the splitting morphisms of q_D^B.

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Proposition

Let D be a WBHA and $(B, f, g) \in |\mathcal{P}roj(D)|$. It holds that:

$$B_D \xrightarrow{i_D^B} B \underset{(B\otimes(\Pi_D^L\circ g))\circ\delta_B}{\overset{(B\otimes g)\circ\delta_B}{\rightrightarrows}} B\otimes D$$

is an equalizer diagram.

$$B\otimes D \underset{\mu_B\circ(B\otimes(f\circ\Pi_D^L))}{\overset{\mu_B\circ(B\otimes f)}{\rightrightarrows}} B \xrightarrow{p_D^B} B_D$$

is a coequalizer diagram.

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Proposition

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Theorem

Let D be a WBHA such that $\exists\ \lambda_D^{-1}$, $(B, f, g) \in |\mathcal{P}roj(D)|$.

It holds that $(B_D, \eta_{B_D}, \mu_{B_D}, \varepsilon_{B_D}, \delta_{B_D})$ *is a WBHA with:*

weak Yang-Baxter op:
$t_{B_D,B_D} = (\varphi_{B_D} \otimes B_D) \circ (D \otimes r_{B_D,B_D}) \circ (\varrho_{B_D} \otimes B_D)$
with $r_{B_D,B_D} = (p_B^D \otimes p_B^D) \circ t_{B,B} \circ (i_D^B \otimes i_D^B) : B_D \otimes B_D \to B_D \otimes B_D$,

$\eta_{B_D} = p_D^B \circ \eta_B, \quad \mu_{B_D} = p_D^B \circ \mu_B \circ (i_D^B \otimes i_D^B),$

$\varepsilon_{B_D} = \varepsilon_B \circ i_D^B, \quad \delta_{B_D} = (p_D^B \otimes p_D^B) \circ \delta_B \circ i_D^B)$

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Index

The braided case

Hypothesis

Let $\mathcal{C}$ be a braided category and H a WHA in $\mathcal{C}$.

Theorem

If $\mathcal{C}$ is a braided category and H a WHA such that $\exists \lambda_H^{-1}$

Then ${}_H^H\mathcal{YD}$ is braided monoidal.

Given $M, N \in {}_D^D\mathcal{YD}$, its product is $M \times N$.
The braid is given by:

$$\tau_{M,N} := p_{N\otimes M} \circ t_{M,N} \circ i_{M\otimes N} : M \times N \to N \times M$$

with $t_{M,N} = (\varphi_N \otimes M) \circ (H \otimes c_{M,N}) \circ (\varrho_M \otimes N) : M \otimes N \to N \otimes M$ and

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The projection theorem in the braided case

Theorem

Let $\mathcal{C}$ be a braided cat., H a WHA such that $\exists \lambda_H^{-1}$,

$(D, u_D, m_D, e_D, \Delta_D, \lambda_D)$ a WHA in ${}_H^H\mathcal{YD}$.

Then $(D, \eta_D, \mu_D, \varepsilon_D, \delta_D, \lambda_D)$ is a WBHA in $\mathcal{C}$. ◂ smash ◂ bosinverse

$$\eta_D = u_D \circ p_L \circ \eta_H, \quad \mu_D = m_D \circ p_{D\otimes D},$$

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and weak Yang-Baxter operator

$$t_{D,D} = (\varphi_D \otimes D) \circ (H \otimes c_{D,D}) \circ (\varrho_D \otimes D).$$

Remark

Moreover: **D is not a Hopf algebra neither a WHA in $\mathcal{C}$.**

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Remark

Moreover: ***D* is not a Hopf algebra neither a WHA in $\mathcal{C}$.**

Weak smash biproduct

Let H be a WHA in $\mathcal{C}$ and $(D, u_D, m_D, e_D, \Delta_D, \lambda_D)$ a WHA in ${}^H_H\mathcal{YD}$.

We define the weak smash biproduct of D and H $(D \times H, \eta_{D\times H}, \mu_{D\times H}, \varepsilon_{D\times H}, \delta_{D\times H}, \lambda_{D\times H})$:

$$\eta_{D\times H} = p_{D\otimes H} \circ (\eta_D \otimes \eta_H),$$

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Let H be a WHA in $\mathcal{C}$ and $(D, u_D, m_D, e_D, \Delta_D, \lambda_D)$ a WHA in ${}^H_H\mathcal{YD}$.

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$$\lambda_{D\times H} = p_{D\otimes H}\circ(\varphi_D\otimes H)\circ(H\otimes c_{H,D})\circ((\delta_H\circ\lambda_H\circ\mu_H)\otimes\lambda_D)\circ(H\otimes c_{D,H}) \circ(\varrho_D \otimes H) \circ i_{D\otimes H}.$$

Weak smash biproduct

Let H be a WHA in $\mathcal{C}$ and $(D, u_D, m_D, e_D, \Delta_D, \lambda_D)$ a WHA in ${}^H_H\mathcal{YD}$.

We define the weak smash biproduct of D and H $(D \times H, \eta_{D\times H}, \mu_{D\times H}, \varepsilon_{D\times H}, \delta_{D\times H}, \lambda_{D\times H})$:

$$\eta_{D\times H} = p_{D\otimes H} \circ (\eta_D \otimes \eta_H),$$

$$\mu_{D\times H} = p_{D\otimes H} \circ (\mu_D \otimes \mu_H) \circ (D \otimes ((\varphi_D \otimes H) \circ (H \otimes c_{H,D}) \circ (\delta_H \otimes D)) \otimes H) \circ (i_{D\otimes H} \otimes i_{D\otimes H}),$$

$$\varepsilon_{D\times H} = (\varepsilon_D \otimes \varepsilon_H) \circ i_{D\otimes H},$$

$$\delta_{D\times H} = (p_{D\otimes H} \otimes p_{D\otimes H}) \circ (D \otimes ((\mu_H \otimes D) \circ (H \otimes c_{D,H}) \circ (\varrho_D \otimes H)) \otimes H) \circ (\delta_D \otimes \delta_H) \circ i_{D\otimes H},$$

$$\lambda_{D\times H} = p_{D\otimes H} \circ (\varphi_D \otimes H) \circ (H \otimes c_{H,D}) \circ ((\delta_H \circ \lambda_H \circ \mu_H) \otimes \lambda_D) \circ (H \otimes c_{D,H}) \circ (\varrho_D \otimes H) \circ i_{D\otimes H}.$$

Weak smash biproduct

Theorem

Let H be a WHA, $(D, u_D, m_D, e_D, \Delta_D, \lambda_D)$ *a Hopf algebra in* ${}_{H}^{H}\mathcal{YD}$*:*

Then the weak smash biproduct $D \times H$ *is a WHA in* $\mathcal{C}$.

The projection theorem in the braided case

Theorem

Let H be a WHA such that $\exists\ \lambda_H^{-1}$, $(B, f, g) \in |\mathcal{P}roj(H)|$. Then:

$(B_H, u_{B_H}, m_{B_H}, e_{B_H}, \Delta_{B_H}, \lambda_{B_H})$ *is a Hopf algebra in* ${}_H^H\mathcal{YD}$. ◂ F

$B \simeq B_H \times H$ *as WHA (* $B_H \times H$ *the weak smash biproduct).*

$$u_{B_H} =: p_H^B \circ f \circ i_L, \quad m_{B_H} := \mu_{B_H} \circ i_{B_H \otimes B_H},$$

$$e_{B_H} := p_L \circ g \circ i_H^B, \quad \Delta_{B_H} := p_{B_H \otimes B_H} \circ \delta_{B_H},$$

$$\lambda_{B_H} := p_H^B \circ ((f \circ g) \wedge \lambda_B) \circ i_H^B.$$

The projection theorem in the braided case

Proposition

Let H be a WHA such that $\exists\, \lambda_H^{-1}$,

$D = (D, u_D, m_D, e_D, \Delta_D, \lambda_D)$ *a Hopf algebra in* ${}^H_H\mathcal{YD}$,

$D = (D, \eta_D, \mu_D, \varepsilon_D, \delta_D, \lambda_D)$ *the WBHA in* $\mathcal{C}$ Go

$D \times H$ *the weak smash biproduct.*

Then:

(i) $(D \times H, \tilde{f} := p_{D\otimes H} \circ (\eta_D \otimes H), \tilde{g} := (\varepsilon_D \otimes H) \circ i_{D\otimes H}) \in |\mathcal{P}roj(H)|$.

G

(ii) $D \simeq (D \times H)_H$ *as WBHA.*

The projection theorem in the braided case

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Let H be a WHA such that $\exists\ \lambda_H^{-1}$,

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$D = (D, \eta_D, \mu_D, \varepsilon_D, \delta_D, \lambda_D)$ the WBHA in $\mathcal{C}$ Go

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The projection theorem in the braided case

Theorem

If H is a WHA such that $\exists\ \lambda_H^{-1}$*:*

There exists a category equivalence $\mathcal{P}roj(H) \simeq \mathcal{HA}({}_H^H\mathcal{YD})$.

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If H is a WHA such that $\exists\ \lambda_H^{-1}$*:*

There exists a category equivalence $\mathcal{P}roj(H) \simeq \mathcal{HA}({}_H^H\mathcal{YD})$.

$$F : \mathcal{P}roj(H) \longrightarrow \mathcal{HA}({}_H^H\mathcal{YD})$$

Go

$$(B, f, g) \longmapsto (B_H, u_{B_H}, m_{B_H}, e_{B_H}, \Delta_{B_H}, \lambda_{B_H})$$

$$\alpha : B \rightarrow B' \longmapsto \alpha_H : B_H \rightarrow B'_H(\text{ factorization of } \alpha \circ i_H^B\ )$$

The projection theorem in the braided case

Theorem

If H is a WHA such that $\exists\ \lambda_H^{-1}$:

There exists a category equivalence $\mathcal{P}roj(H) \simeq \mathcal{HA}({}_H^H\mathcal{YD})$.

▸ Go

$$
\begin{array}{rcl}
G : \mathcal{HA}({}_H^H\mathcal{YD}) & \longrightarrow & \mathcal{P}roj(H) \\
(D, u_D, m_D, e_D, \Delta_D, \lambda_D) & \longmapsto & (D \times H, p_{D\otimes H} \circ (\eta_D \times H), (\varepsilon_D \times H) \circ i_{D\otimes H} \\
r : D \to D' & \longmapsto & r \times H
\end{array}
$$

Projections of weak braided Hopf algebras

Carlos Soneira Calvo

Univ. da Coruña

Workshop on Category Theory. Universidade de Coimbra

9 de julio de 2012

Monoidal structure of ${}^{H}_{H}\mathcal{YD}$

- Base Object: $D_L = Im(\Pi^L_D)$,
- Associativity constraints $\mathfrak{a}_{M,N,P} : M \times (N \times P) \to (M \times N) \times P$

$$\mathfrak{a}_{M,N,P} = p_{(M\times N)\otimes P} \circ (p_{M\otimes N} \otimes P) \circ (M \otimes i_{N\otimes P}) \circ i_{M\otimes(N\times P)}.$$

- Unity constraints:

$$\mathfrak{l}_M = \varphi_M \circ (i_L \otimes M) \circ i_{D_L\otimes M},$$

$$\mathfrak{r}_M = \varphi_M \circ s'_M \circ (M \otimes (\overline{\Pi}_D{}^L \circ i_L)) \circ i_{M\otimes D_L},$$

$$\mathfrak{l}_M^{-1} = p_{D_L\otimes M} \circ (p_L \otimes \varphi_M) \circ ((\delta_D \circ \eta_D) \otimes M),$$

$$\mathfrak{r}_M^{-1} =: p_{M\otimes D_L} \circ (\varphi_M \otimes p_L) \circ (D \otimes s_M) \circ ((\delta_D \circ \eta_D) \otimes M).$$

Monoidal structure of ${}^H_H\mathcal{YD}$

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Projections and WBHA structures

Proposition

Let D be a WBHA and $(B, f, g) \in |\mathcal{P}roj(D)|$.

Then $(B_D, \varphi_{B_D}, \varrho_{B_D}) \in {}_D^D\mathcal{YD}$ *with*

$$\varphi_{B_D} = p_D^B \circ \mu_B \circ (f \otimes i_D^B), \qquad \varrho_{B_D} = (g \otimes p_D^B) \circ \delta_B \circ i_D^B,$$

and:

$$t_{B_D,D} = (g \otimes p_D^B) \circ t_{B,B} \circ (i_D^B \otimes f), \quad t_{D,B_D} = (p_D^B \otimes g) \circ t_{B,B} \circ (f \otimes i_D^B).$$

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Proposition

Let D be a WBHA such that $\exists\ \lambda_D^{-1}$, $(B,f,g) \in |\mathcal{P}roj(D)|$. *Taking:*

$$r_{B_D,B_D} = (p_B^D \otimes p_B^D) \circ t_{B,B} \circ (i_D^B \otimes i_D^B) : B_D \otimes B_D \to B_D \otimes B_D,$$

$$r'_{B_D,B_D} = (p_B^D \otimes p_B^D) \circ t'_{B,B} \circ (i_D^B \otimes i_D^B) : B_D \otimes B_D \to B_D \otimes B_D,$$

the arrow $t_{B_D,B_D} : B_D \otimes B_D \to B_D \otimes B_D$

$$t_{B_D,B_D} = (\varphi_{B_D} \otimes B_D) \circ (D \otimes r_{B_D,B_D}) \circ (\varrho_{B_D} \otimes B_D)$$

is a weak Yang-Baxter operator with

$$t'_{B_D,B_D} = r'_{B_D,B_D} \circ (\varphi_{B_D} \otimes B_D) \circ (t'_{B_D,D} \otimes B_D) \circ (B_D \otimes ((\lambda_D^{-1} \otimes B_D) \circ \varrho_{B_D})).$$

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