A Personal Glance at George's Category Theory

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1974	Diploma	Tbilisi State University
1978	Ph.D.	Tbilisi State University
1992	D.Sc.	St.-Petersburg State University

Georgian Academy of Sciences (since 1975)

McGill, York, Milan, Chicago, Bielefeld, Sydney

Hungarian Academy of Sciences, Trieste, Genova, Wales (at Bangor)

Tours, Louvain-la-Neuve, Littoral (at Calais), Coimbra

Insubria (at Como), Aveiro, IST Lisbon, . . .

University of Cape Town (since 2004)

Major areas of work

Categorical Galois Theory

Descent Theory

Categories for Algebra

Categories for Topology

Categorical Galois Theory

- Galois Theory in categories with inclusions (Proc. Junior Sci. 1974)
- The fundamental theorem of Galois Theory (USSR Sbornik 1989)
- Pure Galois Theory in Categories (J. Algebra 1990)

- Galois Theories (Cambridge 2001, with F. Borceux)

- Categorical Galois Theory: Revision and some recent developments (Potsdam 2001)
- Descent and Galois Theory (Haute Bodeux 2007)

Central extensions – Classically

$A \xrightarrow{\alpha} B$ surjective

$(A, \alpha) \in (\mathbf{Grp} \downarrow B)$ *central extension*

$\iff \ker \alpha \subseteq \text{centre}\,(A)$

(A, α) *trivial* central extension

$\iff (A, \alpha) \cong (K \times B, K \times B \to B)$ with K Abelian

Central extensions – Categorically

$(A, \alpha) \in (\mathbf{Grp} \downarrow B)$ central extension

$\iff$ $\exists\, p : E \to B$ surjective such that

$p^*(A, \alpha)$ trivial:

$$\begin{array}{ccc} E \times_B A & \longrightarrow & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B \end{array}$$

$\iff:$ (A, α) split over (E, p)

Separable extensions – Classically

$A \overset{\alpha}{\leftarrow} B$ in **CR**, B field

Example

$$f \in B[x],\ \deg f \geqslant 1,\ B_f = B[x]/(f) \leftarrow B$$

Facts

$$f = g \cdot h,\ (g,h) = 1 \Longrightarrow B_f \cong B_g \times B_h$$

$$B_{(x-b)^n} \cong B_{x^n}$$

$$f \text{ separable } \iff f = a \cdot \prod_{i=1}^{n} (x - b_i),\ b_i \neq b_j \text{ for } i \neq j$$

$$\iff B_f \cong B \times \ldots \times B$$

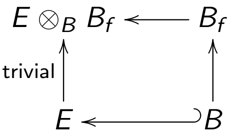
$$\iff B_f \text{ is a } \textit{trivial } B\text{-algebra}$$

Separable extensions – Classically (continued)

If $f \in B[x]$ does not split: $\exists\, E \supseteq B$ such that $f \in E[x]$ splits,

$$E_f \cong E \otimes_B B_f$$

f separable $\iff$ $E \otimes_B B_f$ trivial E-algebra



Separable extensions – Categorically

A separable B-algebra

$\iff$ $\dim_B A < \infty$, $\forall a \in A$: a separable over B

$\iff$ $\exists$ field extension $E \hookleftarrow B$: $E \otimes_B A$ trivial E-algebra

$\iff$: A is split over B

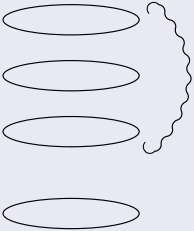
$$\begin{array}{ccc} E \otimes_B A & \longleftarrow & A \\ \uparrow & & \uparrow \alpha \\ E & \longleftarrow & B \end{array}$$

Covering spaces – Classically

$A \xrightarrow{\alpha} B$ local homeomorphism $\iff$ (A, α) étale space over B

Very trivial example

$A_i \subseteq A$ open, $A_i \xrightarrow{\cong} B$



$$\begin{array}{c} A = \bigcup_{i \in I} A_i \quad \text{(disjoint)} \\ \downarrow \alpha \\ B \end{array}$$

Trivial example

$$B = \bigcup_{\lambda \in \Lambda} B_\lambda \text{ (disjoint)}$$

$B_\lambda \subseteq B$ open, $\alpha^{-1}(B_\lambda) \to B_\lambda$ very trivial

Covering spaces – Categorically

(A, α) covering space over B

$\iff$ $\forall\, b \in B\ \exists$ open $V \ni b$ in B: $\alpha^{-1}(V) \to V$ very trivial

$\iff$ $\exists\, E \xrightarrow{p} B$ surjective, étale:

$p^{\star}(A, \alpha)$ trivial

$$\begin{array}{ccc} E \times_B A & \longrightarrow & A \\ \big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ E & \underset{p}{\longrightarrow} & B \end{array}$$

$\iff:$ (A, α) split over (E, p)

The machinery of adjunctions

$$\mathbf{C} \underset{H}{\overset{I}{\rightleftarrows}} \mathbf{X}\ (\bot), \quad \mathbf{C} \text{ with pullbacks, } B \in \mathbf{C}$$

$$(\mathbf{C} \downarrow B) \underset{H^B}{\overset{I^B}{\rightleftarrows}} (\mathbf{X} \downarrow IB)\ (\bot)$$

$$\begin{array}{ccc} B \times_{HIB} HX & \longrightarrow & HX \\ \downarrow \pi_1 & & \downarrow H\varphi \\ B & \xrightarrow[\eta_B]{} & HIB \end{array}$$

$$(A, \alpha) \longmapsto (IA, I\alpha)$$

$$(B \times_{HIB} HX, \pi_1) \longleftarrow\!\!\dashv (X, \varphi)$$

Split objects

(A,α) *trivial* $:\Longleftrightarrow$
$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HIA \\ \alpha\downarrow & & \downarrow HI\alpha \\ B & \xrightarrow[\eta_B]{} & HIB \end{array}$$
pullback

(A,α) *split over* (E,p) $:\Longleftrightarrow$ $p^*(A,\alpha)$ trivial

Example 1

$$\mathbf{Grp} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{AbGrp}$$

α, p surjective, E free

Split objects, continued

Example 2

$$(\mathbf{CR}^{op} \downarrow k)_{\text{fin}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{FinSet}$$

$$A \longmapsto \{\text{minimal non-zero idempotents}\}$$

$$\underbrace{k \times \ldots \times k}_{X \text{ times}} \longleftarrow\!\!\!\mid X$$

$$E \overset{p}{\hookleftarrow} B \quad \text{fields}$$

Split objects, continued

Example 3

$$\mathbf{LCTop} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Set}$$

$$B \longmapsto \pi_0 B$$

$$\text{(discrete)}\ X \longleftarrow\!\shortmid\ X$$

$p : E \to B$ surjective, étale

George's Galois Theorem

$$\mathbf{C} \underset{H}{\overset{I}{\underset{\perp}{\rightleftarrows}}} \mathbf{X} \qquad \mathcal{F} \subseteq \operatorname{mor}\mathbf{C},\ \Phi \subseteq \mathbf{X}\text{: ``\textit{fibrations}''}$$

Hypothesis

- pullbacks of fibrations exist and are fibrations
- isomorphisms are fibrations, closed under composition
- I and H preserve fibrations
- ("*Admissibility*") $\phi : X \to IB$ fibration $\Rightarrow$ $(I(B \times_{HIB} HX) \to IHX \to X)$ isomorphism

Theorem

$p^* : \mathcal{F}(B) \to \mathcal{F}(E)$ monadic $\Rightarrow$ $\mathsf{Spl}(E, p) \simeq \mathbf{X}^{\mathsf{Gal}(E,p)} \bigcap \Phi$

George's Galois Theorem (continued)

$$
\begin{array}{ccccc}
\mathrm{Spl}(E,p) & \longrightarrow & \mathrm{TrivCov}(E) & \simeq & \Phi(IE) \\
\big\downarrow & \text{pullback} & \big\downarrow & \text{(admissible)} & \big\downarrow H^E \\
\Phi(B) & \longrightarrow & \Phi(E) & = & \Phi(E)
\end{array}
$$

$$
\mathrm{Gal}(E,p) = I(\mathrm{Eq}(p)) = (\, I(E \times_B E \times_B E) \rightrightarrows I(E \times_B E) \underset{Ic}{\overset{Id}{\rightleftarrows}} I(E))
$$

$$
\mathbf{X}^{\mathrm{Gal}(E,p)} \ni (A_0, \pi, \xi) \qquad
\begin{array}{ccc}
I(E \times_B E) \times_{(Id,\pi)} A_0 & \xrightarrow{\ \xi\ } & A_0 \\
\big\downarrow & & \big\downarrow \pi \\
I(E \times_B E) & \xrightarrow[Ic]{} & IE
\end{array}
$$

First proof generalizing Magid's Theorem: 1984. In full generality: 1991

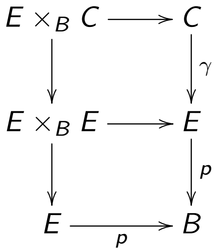
Descent Theory

$p : E \to B$ *effective* (for) *descent*

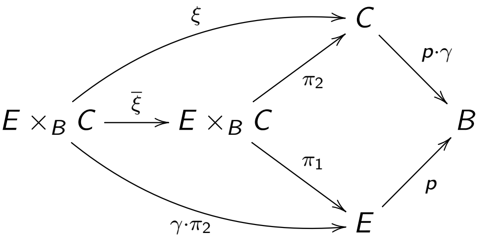
$\iff$ $p_! \dashv p^* : \mathcal{F}(B) \to \mathcal{F}(E)$ monadic

$\iff$ rebuild $\mathcal{F}(B)$ from $\mathcal{F}(E)$ as

$\{(C, \gamma; \xi) : \ \ \xi : E \times_B C \to C, \ 2 \text{ equations}\}$



Equivalent presentation of ξ:



Descent Theory (continued)

$$\mathcal{F}(B) = (\mathbf{Top} \downarrow B)$$

$$(x, y) \in E \times_B E \qquad \begin{array}{ccc} \gamma^{-1}x & \xrightarrow{\xi_{x,y}} & \gamma^{-1}y \\ {\scriptstyle j_{y,x}}\downarrow & & \downarrow{\scriptstyle j_{x,y}} \\ E \times_B C & \xrightarrow{\bar{\xi}} & E \times_B C \end{array} \qquad \begin{array}{c} c \\ \downarrow \\ (x, c) \end{array}$$

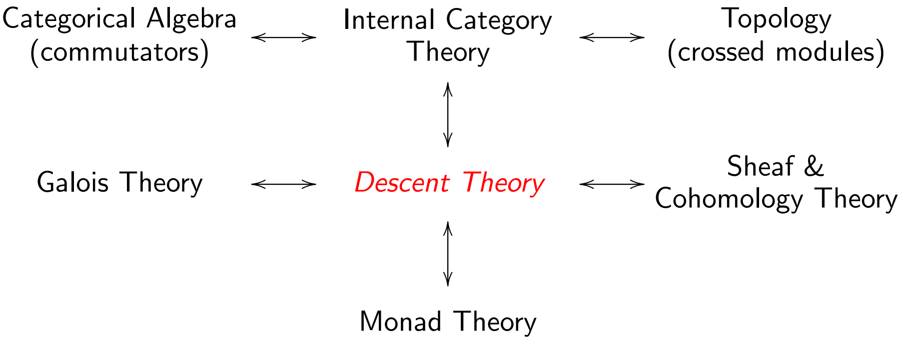
$\xi_{x,x} = \mathrm{id}$, $\xi_{x,z} = \xi_{y,z} \cdot \xi_{x,y}$ $(p(x) = p(y) = p(z))$, *Glueing Condition*

Example

$$E = \sum_{i \in I} U_i \xrightarrow{p} B = \bigcup_{i \in I} U_i \quad (U_i \subseteq B \text{ open})$$

$\xi_{i,j} : \gamma_i^{-1}(U_i \bigcap U_j) \longrightarrow \gamma_j^{-1}(U_i \bigcap U_j)$ satisfying the *Cocycle Condition*

Descent Theory (continued)



Descent Theory (continued)

Two of George's "simple" observations:

- descent $\neq$ effective descent, even in algebra:

$$\{A \in \mathbf{AbGrp} \mid n^2x = 0 \Rightarrow nx = 0\}, \quad p : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$$

- $\mathbf{C} \hookrightarrow \mathbf{D}$ closed under pullbacks,

  $p : E \to B$ in $\mathbf{C}$, effective descent in $\mathbf{D}$. Then:

  p effective descent in $\mathbf{C} \iff \forall\, (A, \alpha) \in (\mathbf{D} \downarrow B)$:

$$p^*(A, \alpha) \in (\mathbf{C} \downarrow E) \Rightarrow (A, \alpha) \in (\mathbf{C} \downarrow B)$$

  $\rightsquigarrow$ Reiterman-T characterization of effective descent morphisms in **Top**

  $\rightsquigarrow$ Clementino-Hofmann characterization of effective descent morphisms in **Top**

Descent Theory (continued)

PreOrd $\cong$ **Alexandroff**, **FinPreOrd** $\cong$ **FinTop**

universal quotient:
(=descent)

$$\begin{array}{ccccc} x & \dashrightarrow & y & & E \\ \big| & & \big| & & \downarrow p \\ u & \longrightarrow & v & & B \end{array}$$

$\uparrow \neq$

effective descent:

$$\begin{array}{ccccc} x & \dashrightarrow & y & \dashrightarrow & z \\ \big| & & \big| & & \big| \\ u & \longrightarrow & v & \longrightarrow & w \end{array}$$

$\uparrow \neq$

triquotient:

$$\begin{array}{ccccccccc} x_n & \dashrightarrow & x_{n-1} & \dashrightarrow & \ldots & \dashrightarrow & x_1 & \dashrightarrow & x_0 \\ \big| & & \big| & & & & \big| & & \big| \\ u_n & \longrightarrow & u_{n-1} & \longrightarrow & \ldots & \longrightarrow & u_1 & \longrightarrow & u_0 \end{array}$$

Semi-Abelian Categories

Mac Lane, Duality for groups, Bull. AMS 1950

“Abelian bicategory” $\rightsquigarrow$ “exact category” (Buchsbaum 1955)
$=$ abelian category

$$\begin{array}{ccc} \mathbf{AbGrp} & \longleftrightarrow & \text{abelian category} \\ \big\downarrow & & \big\downarrow \\ \mathbf{Grp} & \longleftrightarrow & ? \end{array}$$

Old-style generalizations in the realm of pointed/additive categories

Semi-Abelian Categories (continued)

New style approaches 1970 - 1998

- "Barr-exact": finite limits
  pullback stable regular epi-mono factorizations
  equivalence relations are effective

  Tierney's "equation": Barr-exact $+$ additive $=$ Abelian

- "Malcev": from varieties to categories
  (Carboni, Kelly, Lambek, Pedicchio, ...)

Barr-exact $+$ Malcev $\rightsquigarrow$ Commutator theory (Janelidze, Pedicchio ...)

Semi-Abelian Categories (continued)

- "Bourn-protomodular" (Como 1990)

$$\mathrm{Pt}\mathbf{C} = (1 \downarrow \mathbf{C}), \qquad \mathrm{Pt}(B) = \mathrm{Pt}(\mathbf{C} \downarrow B) \qquad \begin{array}{c} E \\ p \downarrow \uparrow s \\ B \end{array} \qquad p \cdot s = 1$$

$h : C \to B$: $\quad h^* : \mathrm{Pt}(B) \to \mathrm{Pt}(C)$ reflects isomorphisms

If $C = 0$: $\quad \ker_B : \mathrm{Pt}(B) \to \mathbf{C}$ reflects isomorphisms

$\iff$ Split Short-Five lemma:

$$\begin{array}{ccccc} \bullet & \rightarrowtail & \bullet & \twoheadrightarrow & \bullet \\ u \downarrow & & w \downarrow & & \downarrow v \\ \bullet & \rightarrowtail & \bullet & \twoheadrightarrow & \bullet \end{array} \qquad u,\ v \text{ iso} \Rightarrow w \text{ iso}$$

- Mac Lane (1950): "ABC extension equivalence theorem"

Semi-Abelian Categories (continued)

$$\begin{aligned}
\text{Semi-Abelian} &= \text{Barr-exact} + \text{Bourn-protomodular} \\
&\quad + \text{finite coproducts} + 0 \cong 1 \\
&= \text{Barr-exact} + \text{semi-additive} \\
\text{Semi-additive} &= \forall B: \ \ker_B : \mathrm{Pt}(B) \to \mathbf{C} \text{ monadic} \\
&\quad + \text{finite coproducts} + 0 \cong 1 \\
\text{Abelian} &= \text{Semi-Abelian} + (\text{Semi-Abelian})^{\mathrm{op}}
\end{aligned}$$

Semi-Abelian Categories (continued)

Examples:

- varieties of Ω-groups, $\{$crossed modules$\}$
- $\mathcal{T}$-**Alg**(**Set**) semi-Abelian $\Rightarrow$ $\mathcal{T}$-**Alg**(**C**) Semi-Abelian
  (finite coproducts granted)
- $(\mathbf{Set}_*)^{\mathrm{op}}$

Pointed naturally-Malcev $\subsetneqq$ protomodular $\subsetneqq$ Malcev

Semi-Abelian Categories (continued)

"Old-style" axioms:

- F. Hofmann (1960):

$$\begin{array}{ccc} & \text{normal} & \\ F & \xrightarrow{\;q\;} & C \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \text{ normal,} \\ E & \xrightarrow[\;p\;]{} & B \\ & \text{normal} & \end{array} \qquad \ker p \leqslant w \Rightarrow w \text{ normal}$$

  $\rightsquigarrow$ protomodular

- p, q, w normal $\Rightarrow$ v normal

  $\rightsquigarrow$ equivalence relations are effective

- pointed, finitely complete, finitely cocomplete
- $p : E \to B$ split epi with sections $s : B \to E$ $\Rightarrow \ker p + B \to E$ normal epi
- normal epis pullback stable

} "homological"

- image of normal mono by normal epi is normal