Pseudogrupoids and *hoc genus omne* in universal algebra

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First Slide

The contents of the first slide will appear on the second slide. And it is much superior to any of Epimenides', Gödel's or Tarski's tricks; because it is

TRUE

First slide-cnt'd

THANK YOU, GEORGE

When back home, slap your wife. You do not need to know why; she does.

(Old Sicilian Philosophy)

Pseudogrupoids-1

• R, S congruence relations on an algebra A
•$R\square S$: the subalgebra of $A \times A \times A \times A$ containing the quadruples (x, y, t, z) such that $x\,R\,y, x\,S\,t, z\,R\,t, z\,S\,y$:

$$\begin{pmatrix} x & t \\ y & z \end{pmatrix}$$

horizontal (resp. vertical) elements related by R (resp. by S).

Pseudogrupoids-2

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A homomorphism $m : R\Box S \longrightarrow A$ is called a *pseudogroupoid on* R, S, if

(A) $x \, S \, m(x, y, t, z) \, R \, z$;

(B) $m(x, y, t, z) = m(x, y, t', z)$ (i.e. m does not depend on the third variable);

(C1) $m(x, x, t, z) = z$;

(C2) $m(x, y, t, y) = x$;

(D) $m(m(x_1, x_2, y, x_3), x_4, t, x_5) = m(x_1, x_2, t, m(x_3, x_4, z, x_5))$,

whenever m is defined [...]for (A), (B), (C1), C(2) and for
$x_1 \ R \ x_2, y \ R \ x_3 \ R \ x_4, t \ R \ x_5 \ R \ z$; and
$t \ S \ x_1 \ S \ y, x_2 \ S \ x_3 \ S \ z, x_4 \ S \ x_5$ for (D).

This title does not exist-1

Axiom (B) suggests a variant: forget about the third coordinate.

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Axiom (B) suggests a variant: forget about the third coordinate. Define $R \llcorner S \subseteq A \times A \times A$ by: $(x, y, z) \in R \llcorner S$ iff there exists $t \in A$ such that $(x, y, t, z) \in R \Box S$. $R \llcorner S$ is trivially a subalgebra of $A \times A \times A$. Thus to represent such a triple, we can use :

$$\begin{pmatrix} x & (t) \\ y & z \end{pmatrix}$$

implying that (t) is needed, but is also is damned.

This title does not exist-2

A homomorphism $h : R \llcorner S \longrightarrow A$ is called a *paragrouopoid on* R, S if

(A') $x \; S \; h(x, y, z) \; R \; z$;

(C'1) $h(x, x, z) = z$;

(C'2) $h(x, y, y) = x$;

(D') $h(x_1, x_2, h(x_3, x_4, x_5)) = h(h(x_1, x_2, x_3), x_4, x_5)$,

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Theorem

There is a pseudogroupoid m on R, S iff there is a paragroupoid h on R, S.

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There is a pseudogroupoid m on R, S iff there is a paragroupoid h on R, S.

• Warning: George does not like this at all; that's why these slides have no title.

Ideal determined varieties

• A variety **C** of universal algebras pointed at 0 is *ideal determined* (shortly, an ID variety) if congruences in **C**

1. are determined by their 0-classes and
2. they are 0-permutable

(meaning that if $0/R = 0/S$, then $R = S$, and if $0\ R\ a\ S\ b$ then for some $c, 0\ S\ c\ R\ b$).

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- **C** is ideal determined iff for some $n \geq 0$ there are binary terms $s, d_1, \ldots, d_n$ such that
  (a) s is a subtraction, i.e. the identities $s(x,x) = 0, s(x,0) = x$ hold in **C** ;
  (b) $d_1, \ldots d_n$ internalize equality, namely $x = y$ iff $d_i(x,y) = 0$ for all $i = 1, \ldots, n$.

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• For a congruence R of $A \in$**C**, one has $a \; R \; b$ iff $d_i(a, b) \; R \; 0$ for all $i = 1, \ldots, n$.

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• Congruence lattices of algebras in an ID variety are modular.

Pseudogrupoids in ID varieties

Theorem

Let $\mathbf{C}$ *be an ID variety,* $A \in \mathbf{C}$ *,* R, S *be congruence relations of* A*. A homomorphism* $g : R \Box S \longrightarrow A$ *is a pseudogroupoid on* R, S *iff the following hold:*

1. $g(x, x, x, x) = x;$
2. $g(x, 0, 0, 0) = x;$
3. $g(0, 0, 0, x) = x;$
4. $g(0, 0, x, x) = x,$

when defined, namely for all $x \in A$ *for (1);* $0\ R\ x\ S\ 0)$ *for (2) and (3);* $(0\ S\ x)$ *for (4).*

Proof

One direction is trivial.

• Assuming (1)-(4) we have to show that g is a pseudogroupoid. Assume the binary terms $s, d_1, \ldots, d_n$ satify requirements (a), (b) above. Consider ideals $I = 0/R, J = 0/S$. First prove some consequences of axioms (1)-(4)(in brackets, the range of the variables):

(5) $g(x, x, 0, 0) = 0 \quad (x \in J)$;

(6) $g(x, x, z, z) = z \quad (x\, S\, z)$;

(7) $g(0, 0, x, 0) = 0 \quad (x \in I \cap J)$;

(8) $g(x, x, 0, x) = x \quad (x \in I \cap J)$;

(9) $g(0, x, x, x) = 0 \quad (x \in I \cap J)$;

(10) $g(0, x, 0, x) = 0 \quad (x \in J)$;

(11) $g(x, y, x, y) = x \quad (x\, S\, y)$;

(12) $g(0, 0, t, z) = z \quad (t\, R\, z, t \in J, z \in J)$.

Proof, cnt'd

- for instance: to prove (9), use (1) and (2):

$$g(0,x,x,x) = g(s(x,x), s(x,0), s(x,0), s(x,0)) =$$
$$= s(g(x,x,x,x), g(0,0,0,x)) = s(x,x) = 0.$$

- For (12), first notice that for $i = 1, \ldots, n$, $d_i(t,z) \in J$; then by (6) and (7):

$$d_i(g(0,0,t,z),z) = d_i(g(0,0,t,z), g(0,0,z,z))) =$$
$$= g(0,0,d_i(t,z),0) = 0.$$

- Next got to the axioms; for instance, to verify (C2) for g: use axiom (B) just verified, and apply (11):

$$g(x,y,t,y) = g(x,y,x,y) = x.$$

Variations on axioms

•A homomorphism $g : R\Box S \longrightarrow A$ is a pseudogroupoid on R, S iff the following hold:

1 $g(x, x, x, x) = x$;

2' $g(x, 0, x, 0) = x$;

3' $g(0, 0, x, 0) = 0$;

4' $g(0, 0, x, x) = x$.

- The real role of axiom 1 is to ensure that g is surjective on A.
- Anybody fit to compact these axioms?

The commutator

- A principal result of [G.J.-Pedicchio (2001)] :

Theorem

In a congruence modular variety, if R, S *are congruences of* A, *then* $[R, S] = \Delta_A]$ *iff there is a pseudogrupoid on* R, S.

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▶ **C** be any (pointed) variety; a term $t(\vec{x}, \vec{y}, \vec{z})$ in distinct tuples of variables $\vec{x} = x_1, \ldots, x_m; \vec{y} = y_1, \ldots, y_n; \vec{z} = z_1, \ldots, z_p$, is a *commutator term in* $\vec{y}, \vec{z}$ if the identities

$$t(\vec{x}, \vec{0}, \vec{z}) = 0, \quad t(\vec{x}, \vec{y}, \vec{0}) = 0$$

hold in $\mathbb{C}$. For subalgebras X, Y of $A \in \mathbb{C}$, their *commutator* $[X, Y]$ is defined:

$$\{t(\vec{a}, \vec{u}, \vec{v}) | t \in CT(\vec{y}, \vec{z}), \vec{a} \in A, \vec{u} \in X, \vec{v} \in Y\}.$$

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▶ it is a normal subalgebra (i.e. it is a congruence class and a subalgebra) of A, it is preserved under surjective homomorphisms, and it depends on A but not on the ID variety to which A belongs.

The commutator in ID varieties-1

In an ID variety, because of congruence modularity, we have the usual modular commutator $[R, S]$.

- $[0/R, 0/S]$ is a congruence class of $[R, S]$, namely

$$[0/R, 0/S] = 0/[R, S].$$

(Gumm-$\sim$ [1984].)

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Let R, S be congruences of an algebra A in an ideal determined variety. Then $[0/R, 0/S] = 0$ iff there is a pseudogroupoid on R, S iff there is a paragrupoid on R, S.

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The shortest direct proof of $\Leftarrow$:assume g is a pseudogroupoid on R, S and $I = 0/R, J = 0/S$. Let $t(x, y, z)$ be a commutator term let $a \in A, b \in I, c \in J$. Then

$$\begin{aligned} t(a, b, c) &= t(g(a, a, a, a), g(b, 0, b, 0), g(0, 0, c, c)) = \\ &= g(t(a, b, 0), t(a, 0, 0), g(a, b, c), t(a, 0, c)) = \\ &= g(0, 0, t(a, b, c), 0) = 0. \end{aligned}$$

Thus $[0/R, 0/S = 0]$.

The commutator in ID varieties-2

Three trivialities:
1 B, C algebras of the same signature; a subalgebra F of $B \times C$ is a *functional subalgebra* of $B \times C$ if it is functional: $(b, c), (b, c') \in F \Rightarrow c = c'$.Such an F is called a *functional relation from B to C*.

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2 $dom(F) =: \{b \in B | \exists c (b, c) \in F\}$ is a subalgebra of B; the restriction $\dagger F = \{(b, c) \in F | b \in dom(F)\}$ is a functional subalgebra of $dom(F) \times C$ which is (the graph of) a mapping $\dagger F : dom(F) \longrightarrow B$ and which is a homomorphism. (Every homomorphism g from a subalgebra S of B into C arises in this way).

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3 Any intersection of functional subalgebras of $B \times C$ is a functional subalgebra. The following are equivalent for any functional subset $H \subseteq B \times C$.

(i) There is a functional subalgebra $F \subseteq B \times C$ such that $F \supseteq H$.

(ii) The subalgebra $H_{B \times C}$ generated in $B \times C$ by H is functional.

The commutator in ID varieties-3

- Let **C** be an ID variety, $A \in$**C** , R, S congruence relations of A; $I = 0/R, J = 0/S$, and let $H(R,S) \subseteq A \times A \times A \times A \times A$ be the union of the following sets of 5-tuples:

  $\{(a,a,a,a,a) | a \in A\}$;

  $\{(a,0,a,0,a) | a \in I\}$;

  $\{(0,0,a,0,0) | a \in I \cap J\}$

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  Notice that $H(R,S)$ is functional in $(A \times A \times A \times A) \times A$.

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Corollary

Let **C** *be an ID variety,* $A \in$**C** , R, S *be congruence relations of* A*. Then* $[R,S] = \Delta_A$ *iff the subalgebra generated in* $A \times A \times A \times A \times A$ *by* $H(R,S)$ *is functional in* $(A \times A \times A \times A) \times A$*.*

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- Ideal determined categories have been invented [G.Janledze-Marki-Tholen-∼(CahiersTGDC2010)]: *extend the above to ideal determined categories*

Beyond Ideal Determinacy and perspectives-1

- A *clot* in a (pointed) algebra A is a subalgebra K such that whenever $t(\vec{x}, \vec{y})$ is a term, and for $\vec{a} \in A, t(\vec{a}, \vec{0}) = 0$, then for $\vec{k} \in K, t(\vec{a}, \vec{k}) \in K$. Equivalently (Agliano'-$\sim$ (J. Austral.M.S.1992)) iff there is a reflexive subalgebra S of $A \times A$ such that $K = 0/S =: \{k \in A | (0, k) \in S\}$.

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• A variety **C** is ideal determined and congruence permutable iff it is *clot determined*: when S, S' are reflexive subalgebras of $A \times A$, if $0/S = 0/S' \Rightarrow S = S'$. A notion of *clot determined categories* should be quite within reach ....

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•*Extend the previous remarks on pseudogrupoids and the commutator to clot determined varieties and categories*

Beyond Ideal Determinacy and perspectives-2

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The commutator in semiabelian categories is dealt with in [Gran-G.Janelidze-$\sim$ (to appear, 2012)], *but not yet via pseudogrupoids*

Beyond Ideal Determinacy and perspectives-3

Stepping out from the pointed case

- **(1)** We have also *cosets* in universal algebra [Agliano'-$\sim$(J.Algebra,1987)]: a *coset* in $A \in \mathbf{C}$ is a subset $K \subseteq A$ such that whenever an identity

$$t(x_1, \ldots, x_m, z, \ldots, z) = z$$

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Variety $\mathbf{C}$ is *coset determined* if every coset is a congruence class for exactly one congruence: then it turn out this happens iff the variety is congruence regular (congruences with a class in common coincide) and congruence permutable.

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Extend all of the above to coset determined categories.

- **(2)** Ideals, clots and the commutator can be extended to general varieties with many constants [$\sim$ (TAC, 2012)]. *What are pseudogrupoids here?*

None of the above, but in this section

A curious remark on some semiabelian varieties

- A 1- semiabelian variety is a variety satisfying the laws:

$$m(x, x) = 0$$
$$p(y, d(x, y)) = x$$

for some binary terms m, p : you have "both addition and subtraction". (A.k.a "Bidual Algebren in German. Considered by [Słominski (Fund. Math.1960)]. In fact, all we say is implicit in the masterpiece [Mal'tsev(Mat.Sb.1954)])

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• Johnstone showed that not all semiabelian varieties are 1-semiabelian.

• A *translation* of an algebra A is a unary term function

None of the above, but in this section

A curious remark on some semiabelian varieties

- A 1- semiabelian variety is a variety satisfying the laws:

$$m(x, x) = 0$$
$$p(y, d(x, y)) = x$$

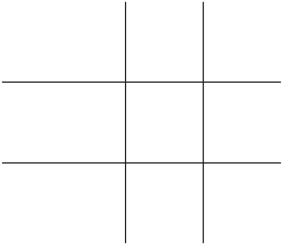
for some binary terms m, p : you have "both addition and subtraction". (A.k.a "Bidual Algebren in German. Considered by [Słominski (Fund. Math.1960)]. In fact, all we say is implicit in the masterpiece [Mal'tsev(Mat.Sb.1954)])

• Johnstone showed that not all semiabelian varieties are 1-semiabelian.

• A *translation* of an algebra A is a unary term function

- A variety is 1-semiabelian iff the group of invertible translations over every algebra in $\mathbf{C}$ is transitive.

THE END



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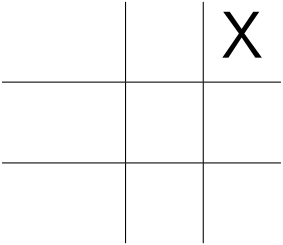
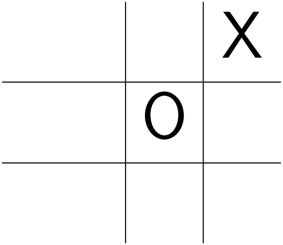


(I was X, playing as an idiot, and I lost -I hope you gained)

THE END



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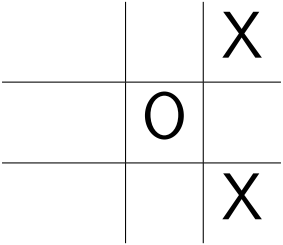

(I was X, playing as an idiot, and I lost -I hope you gained)

THE END

		X
	O	O
		X

(I was X, playing as an idiot, and I lost -I hope you gained)

THE END

		X
X	O	O
		X

(I was X, playing as an idiot, and I lost -I hope you gained)

THE END

		X
X	O	O
	O	X

(I was X, playing as an idiot, and I lost -I hope you gained)

THE END

	X	X
X	O	O
	O	X

(I was X, playing as an idiot, and I lost -I hope you gained)

THE END

O	X	X
X	O	O
	O	X

(I was X, playing as an idiot, and I lost -I hope you gained)

THE END

O	X	X
X	O	O
end	O	X

(I was X, playing as an idiot, and I lost -I hope you gained)