The Algebra and Geometry of Networks

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in collaboration with

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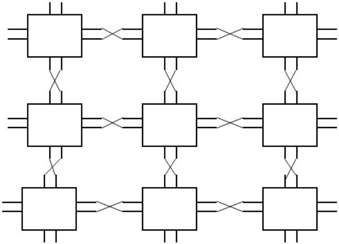
Talk presented at the Workshop in honour of George Janelidze
University of Coimbra

11th July 2012

Introduction

We will describe two kinds of networks of interconnected components:

- Tangled circuits - networks in which the tangling of the connecting "wires" is represented;
- Networks with state - networks in which the tangling is ignored but in which the state of the components is represented.



PART I: Tangled circuits Rosebrugh, Sabadini, Walters, Tangled circuits, arXiv:1110.0715 (2011)

A commutative Frobenius algebra in a braided monoidal category consists of an object G and four arrows $\nabla : G \otimes G \to G$, $\Delta : G \to G \otimes G$, $n : I \to G$ and $e : G \to I$ making (G, ∇, e) a monoid, (G, Δ, n) a comonoid and satisfying the equations

$$(1_G \otimes \nabla)(\Delta \otimes 1_G) = \Delta\nabla = (\nabla \otimes 1_G)(1_G \otimes \Delta) : G \otimes G \to G \otimes G$$

$$\nabla\tau = \nabla : G \otimes G \to G$$

$$\tau\Delta = \Delta : G \to G \otimes G$$

where τ is the braiding.

A monoidal graph M consists of two sets M_0 (vertices or wires) and M_1 (edges or components) and two functions $dom : M_1 \to M_0^*$ and $cod : M_1 \to M_0^*$ where M_0^* is the free monoid on M_0.

Given a monoidal graph M the free braided strict monoidal category in which the objects of M are equipped with commutative Frobenius algebra structures is called $\mathbf{TCircD}_M$. Its arrows are called tangled circuit diagrams, or more briefly circuit diagrams.

Tangled circuits

If M is the monoidal graph $\{R : G \otimes G \to G \otimes G\}$ then the following are distinct tangled circuits $G \otimes G \to G \otimes G$:

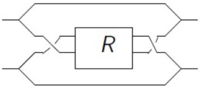


Figure : $(\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(1 \otimes R \otimes 1)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)$

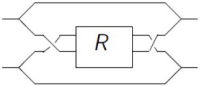
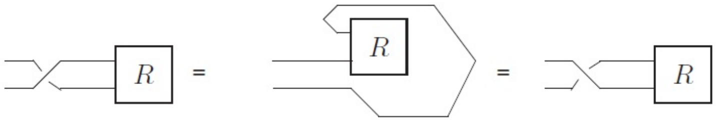


Figure : $(\nabla \otimes \nabla)(1 \otimes \tau^{-1} \otimes 1)(1 \otimes R \otimes 1)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)$

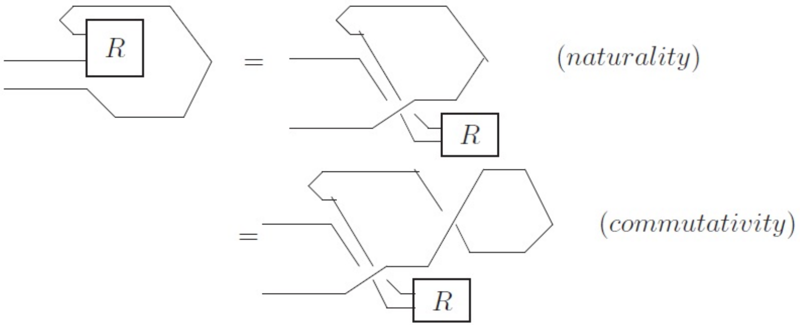
Equality of tangled circuits

We claim that the following three circuits are equal. It clearly suffices to verify the first equation.

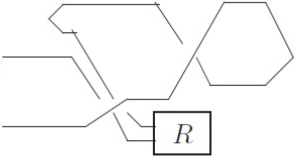


Equality of tangled circuits

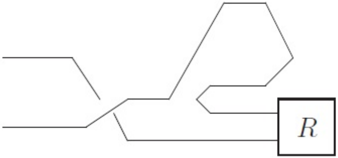
Proof of equality:



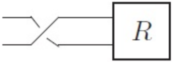
Equality of tangled circuits

= 

(*commutativity*)

= 

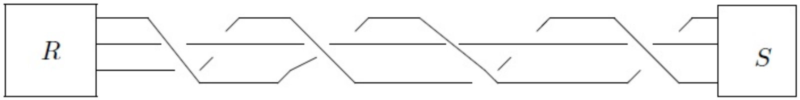
(*naturality*)

= 

(*duality*)

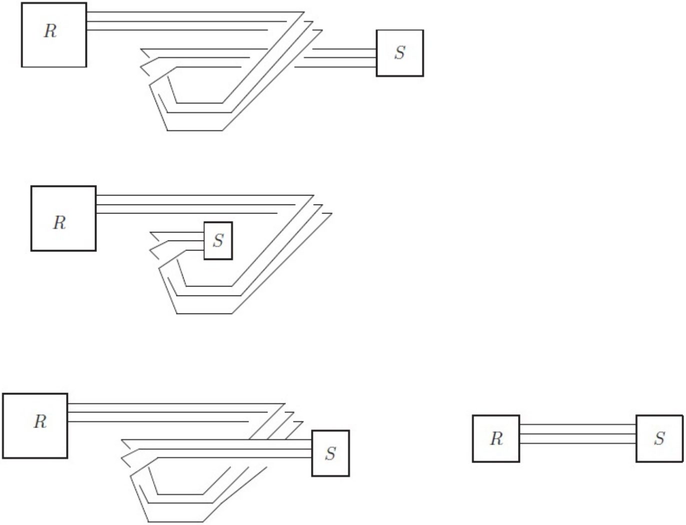
Dirac Belt Trick

Two whole twists can be unwound without rotating the ends - this is called Dirac's belt trick.



Dirac Belt Trick

Rough indication of steps of proof - the two complete twists are equal to



Proving tangled circuits distinct

To prove two circuits distinct it suffices to find a braided monoidal category with a commutative Frobenius algebra in which the two circuits are distinct. Given any group G there is an interesting such category which may be thought of as tangled relations $\mathbf{TRel}_G$.

Proving tangled circuits distinct

Let G be a group. The objects of $\mathbf{TRel}_G$ are the formal powers of G, and the arrows from G^m to G^n are relations R from the set G^m to the set G^n satisfying:

1) if $(x_1, ..., x_m)R(y_1, ...y_n)$ then also for all g in G $(g^{-1}x_1g, ..., g^{-1}x_mg)R(g^{-1}y_1g, ..., g^{-1}y_mg)$,
2) if $(x_1, ..., x_m)R(y_1, ...y_n)$ then $x_1...x_m(y_1...y_n)^{-1} \in Z(G)$ (the center of G).

Composition and identities are defined to be composition and identity of relations.
It is straightforward to verify that $\mathbf{TRel}_G$ is a category.

Proving tangled circuits distinct

We introduce some useful notation. Write $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_n)$, and so on. Write $\overline{x} = x_1 x_2 ... x_m$ and for g, h in G, as $g^h = hgh^{-1}$. For g in G write $x^g = (x_1^g, x_2^g, ..., x_m^g)$. Thus, $(\overline{x})^g = \overline{x^g}$, and of course for any x, y in $G^m \times G^n$, $x^g y^g = (xy)^g$ where we write xy for $(x_1, ..., x_m, y_1, ..., y_n)$.
Then $\mathbf{TRel}_G$ is a braided strict monoidal category with tensor defined on objects by $G^m \otimes G^n = G^{m+n}$ and on arrows by product of relations. The twist

$$\tau_{m,n} : G^m \otimes G^n \to G^n \otimes G^m$$

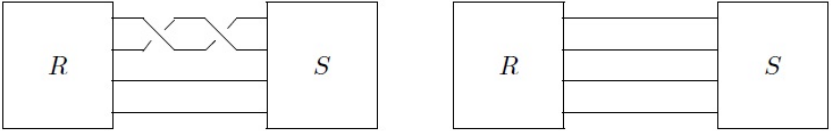
is the functional relation

$$(x, y) \sim (y^{\overline{x}}, x)$$

The arrow $\nabla : G \otimes G \to G$ is the functional relation of multiplication of the group.

Proving tangled circuits distinct

The following two circuits can be shown to be distinct by looking in $TRel_{S_3}$:



In fact with four or more strings and n twists on the first two strings as above we obtain always distinct tangled circuits. With two or three strings there are only finitely many tangled circuits of this type (blocked braids) (Davide Maglia).

PART II: Networks with state

For many (most) purposes the tangling is not of interest. What is of interest is that the components have state and behaviour. So we now consider symmetric monoidal categories generated by a monoidal graph M with Frobenius algebra structures on the objects, but now also with the additional axiom

$$\nabla\Delta = 1.$$

The geometry of this algebra is very simple: Let *MonGraph* be the category of monoidal graphs. Then the symmetric monoidal categories generated by a monoidal graph M with separable algebra (Frobenius + $\nabla\Delta = 1$) structures on the objects is the full subcategory of $Cospan(MonGraph/M)$ restricted to the discrete monoidal graphs - which category we call $Csp(MonGraph/M)$.

Rosebrugh, Sabadini, Walters: Generic commutative separable algebras and cospans of graphs, TAC 2005

Networks with state

Example: The cospan of monoidal graphs corresponding to the expression

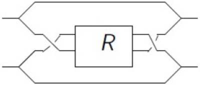
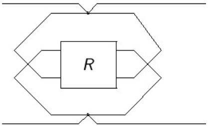


Figure : $(\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)(1 \otimes R \otimes 1)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)$

is



The monoidal graph in the centre has one component labelled R and two wires both labelled G. The left hand leg of the cospan is indicated by the input wires, and the right hand by the output wires.

Networks with state

How does this allow us to describe networks with state? As follows: the monoidal graph M may be the underlying monoidal graph of a monoidal category, for example $Span(Graph)$. So the objects of the monoidal graph may have states and transitions. But there is another fundamental fact. The objects of $Span(Graph)$ have separable algebra structures and hence there is an induced structure-preserving functor

$$Csp(MonGraph/|Span(Graph)|) \longrightarrow Span(Graph),$$

which happens to be obtained by calculating a limit in $Graph$.

Rosebrugh, Sabadini, Walters: Calculating colimits compositionally, LNCS 5065, 2008

Networks with state

Consider the functor

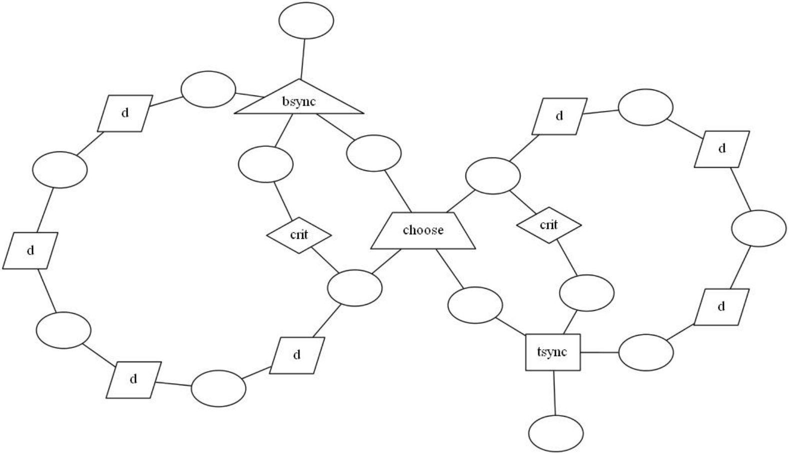
$$limit : Csp(MonGraph/|Span(Graph)|) \longrightarrow Span(Graph).$$

An arrow in the domain may be thought of as either (i) an expression in components with state (the compositional view of a system) or (ii) a network (cospan of monoidal graphs) labelled in graphs. The functor *limit* yields the global states and transitions of the system.

The two points of view of the domain yield two methods of calculating the global states and transitions of a system: (i) by evaluating the expression in the domain and taking a limit, or (ii) evaluating the expression in the codomain.

The first method is much more efficient than the second in finding single paths in the system, and Filippo Schiavio has written a program for doing that, with a graphical output. We present an example of such a calculation for a simple mutual exclusion protocol.

Networks with state



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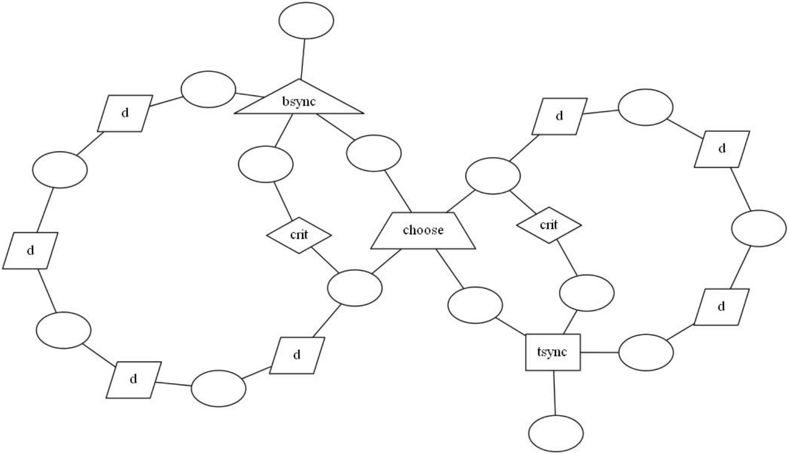

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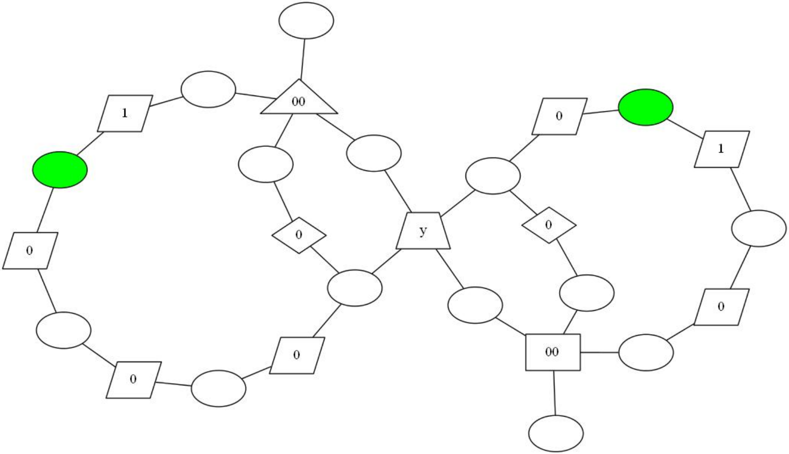

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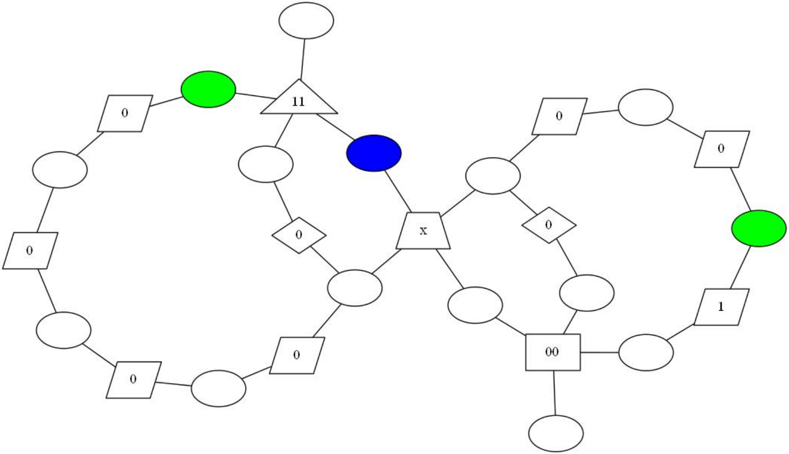

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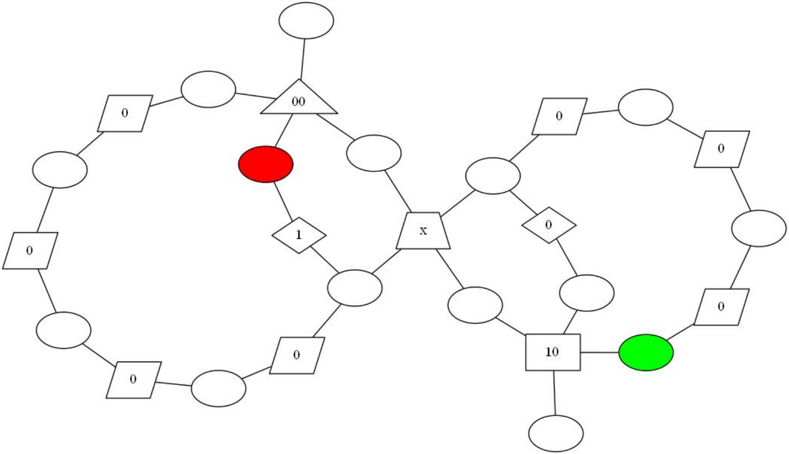

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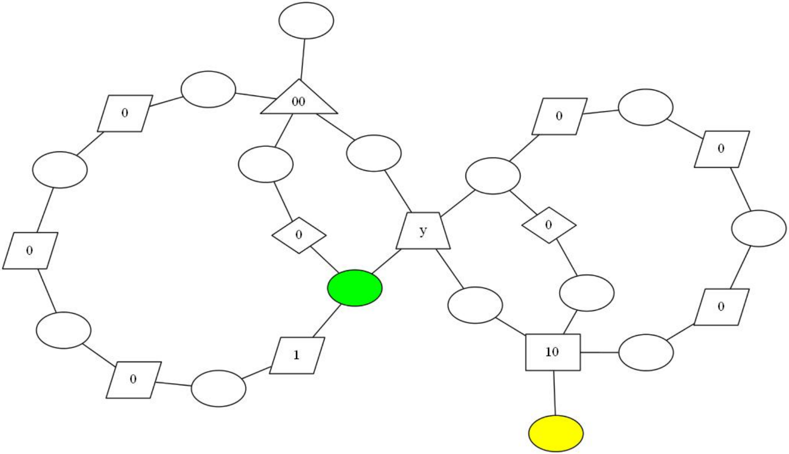

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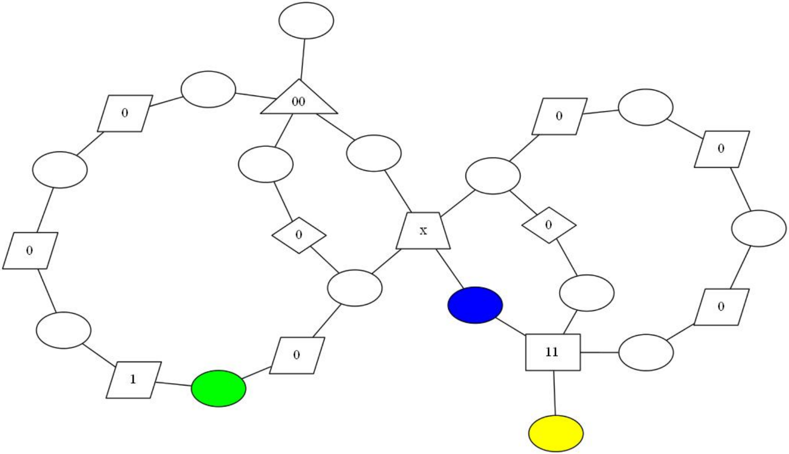

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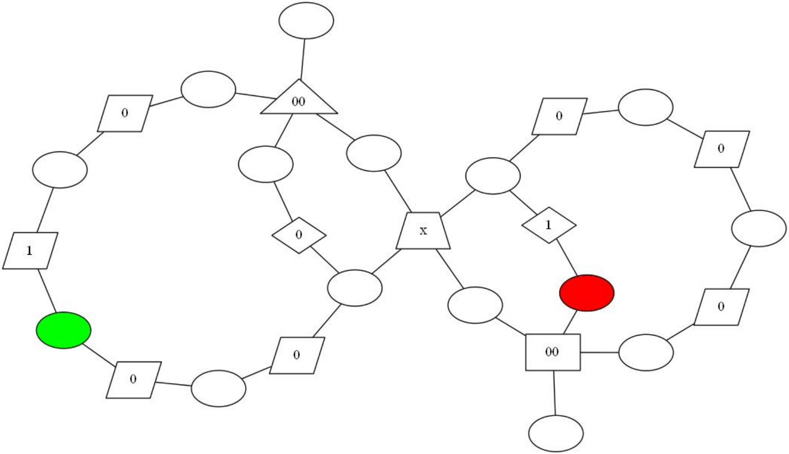

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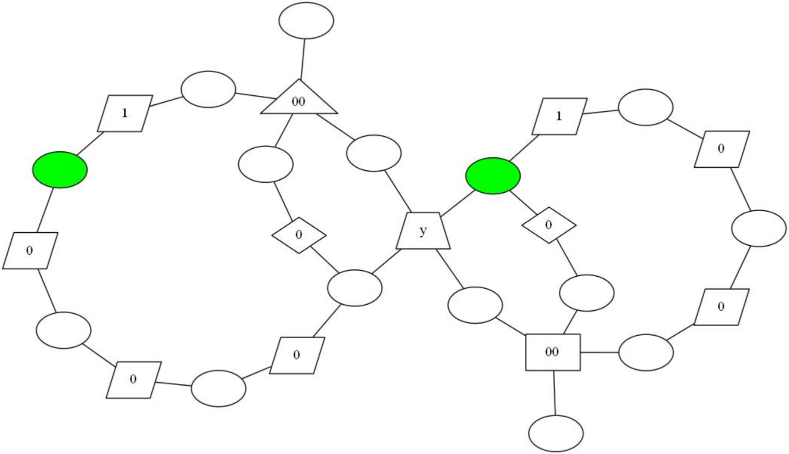

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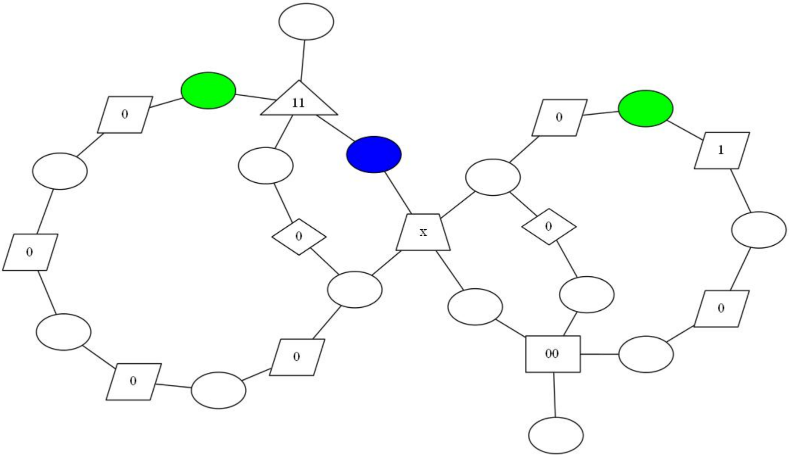

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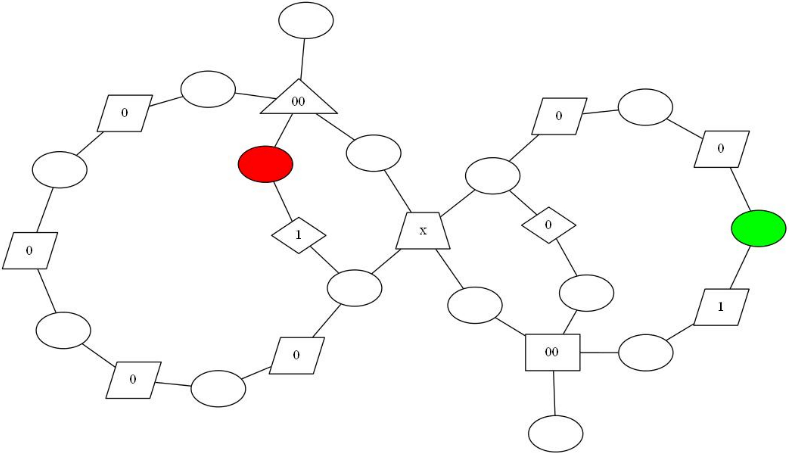

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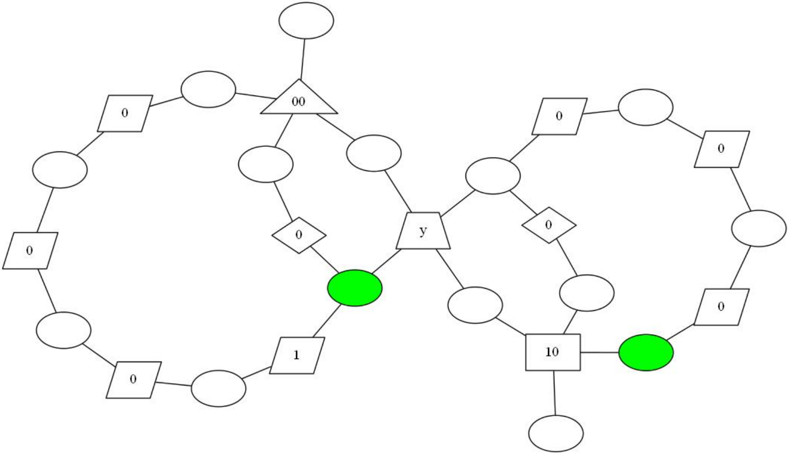

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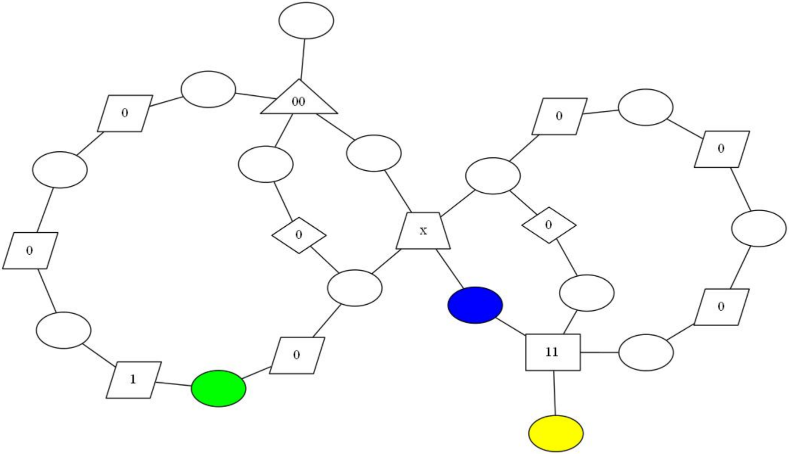

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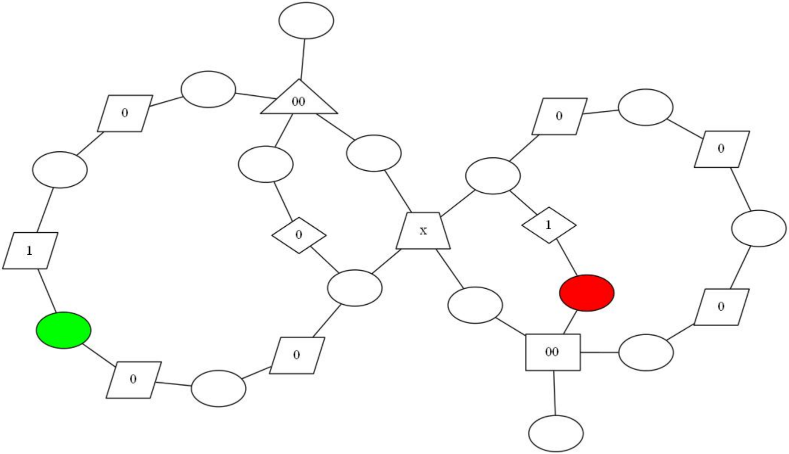

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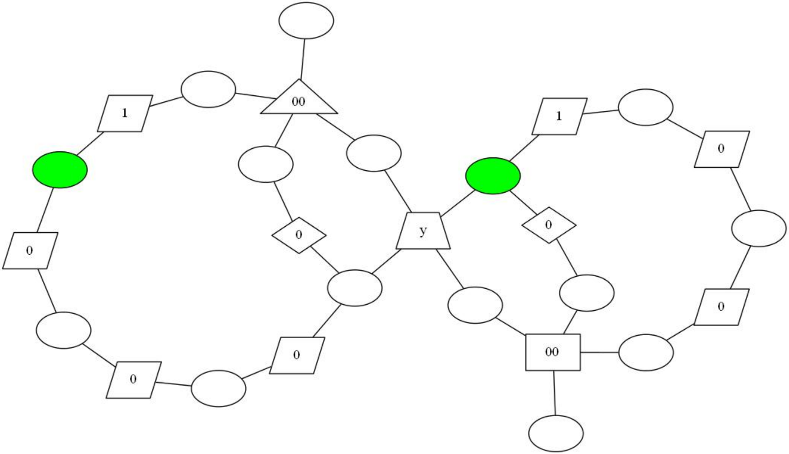

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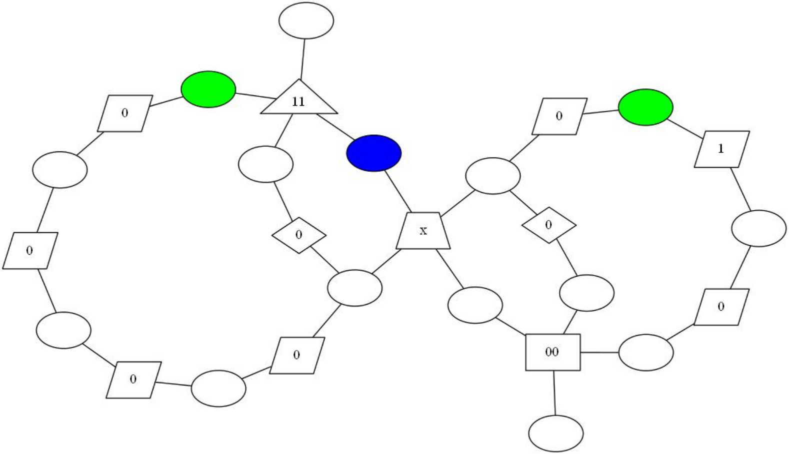

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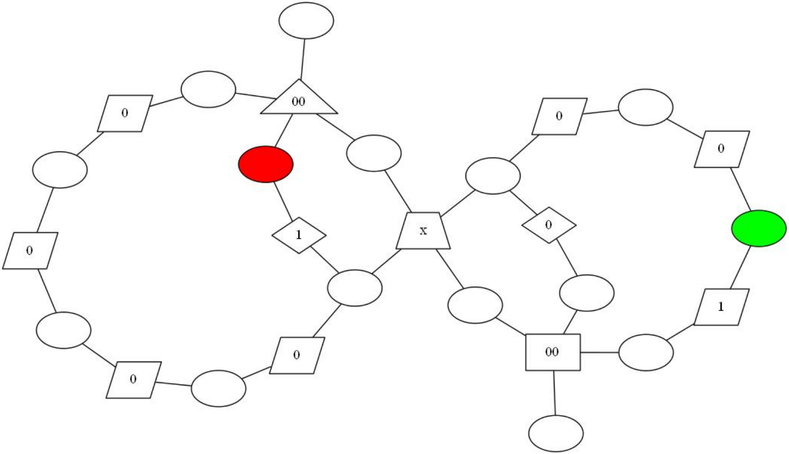

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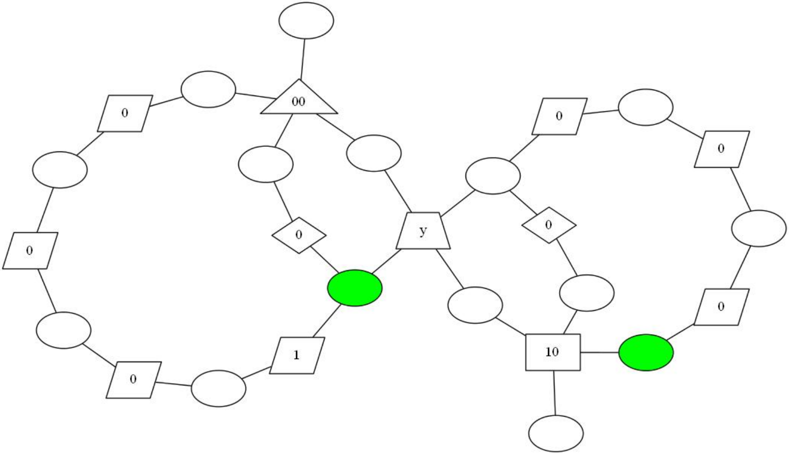

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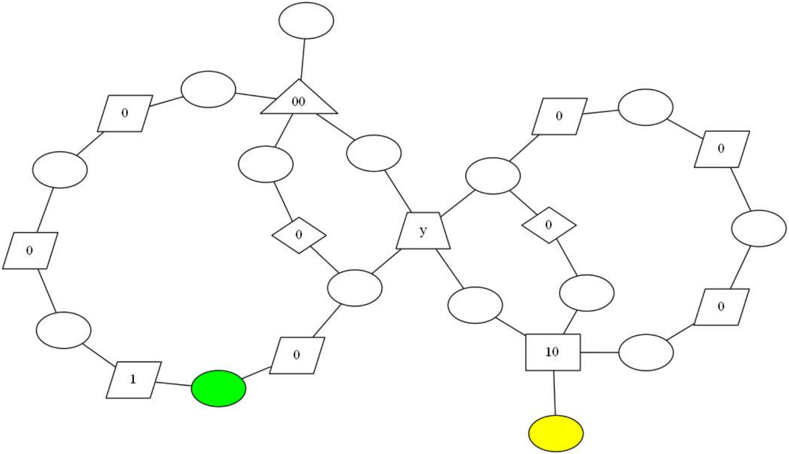

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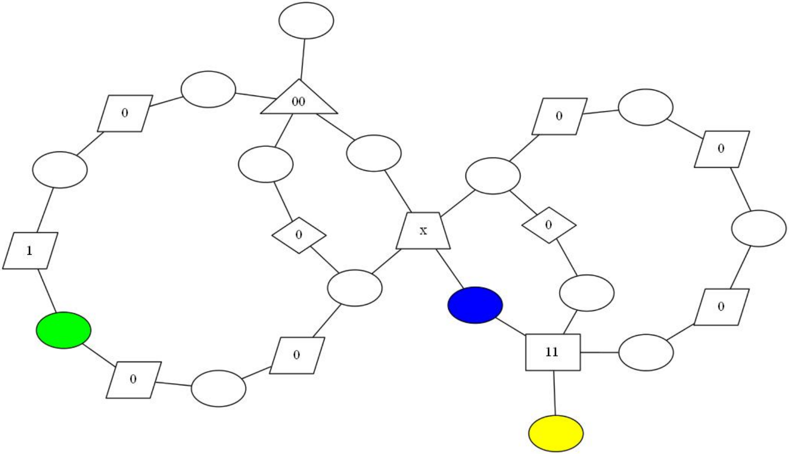

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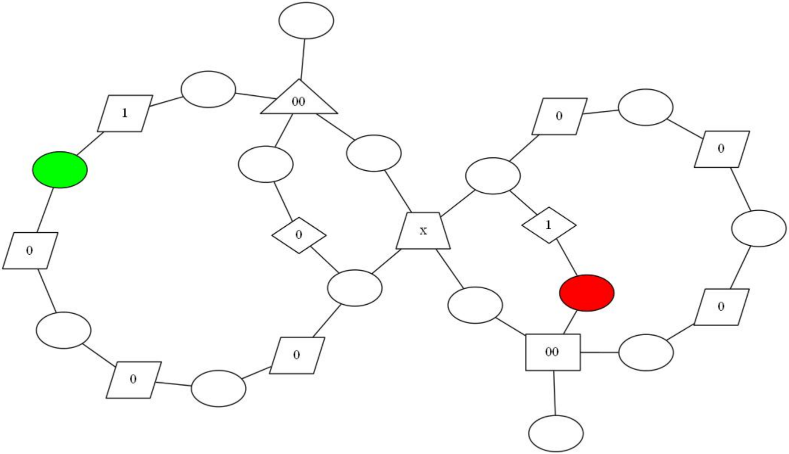

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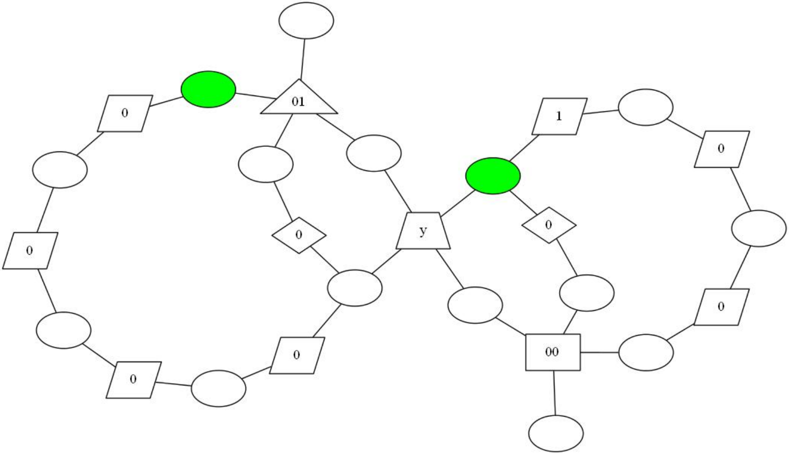

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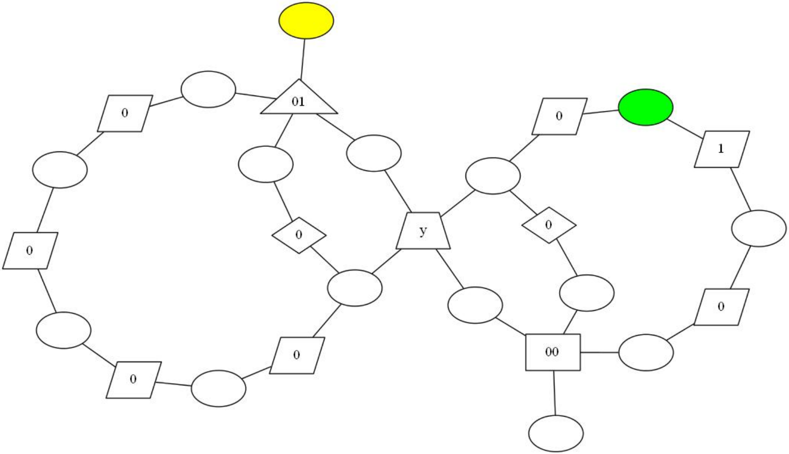

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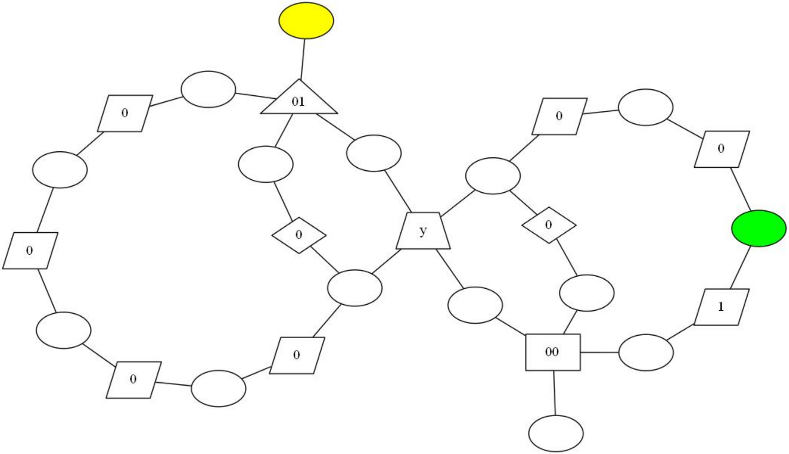

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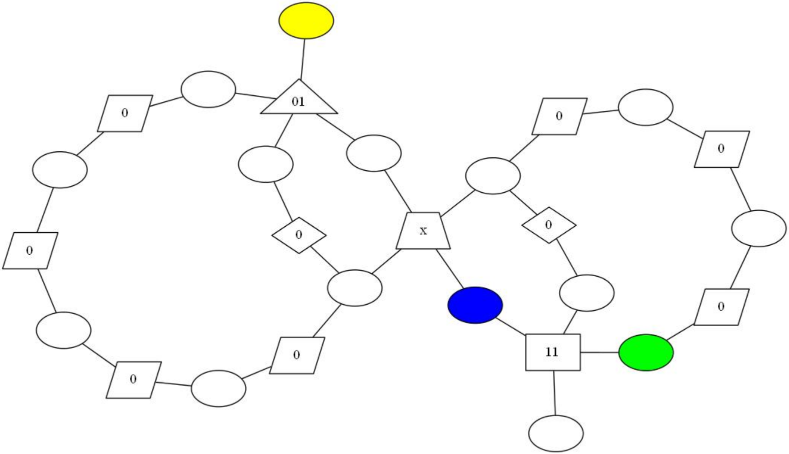

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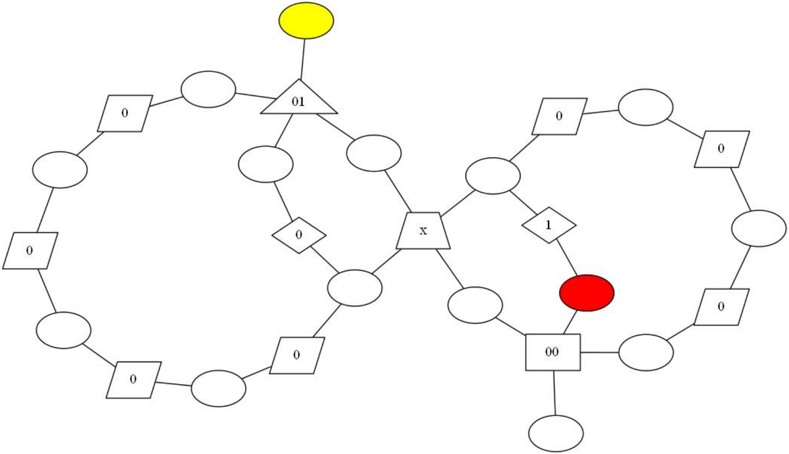

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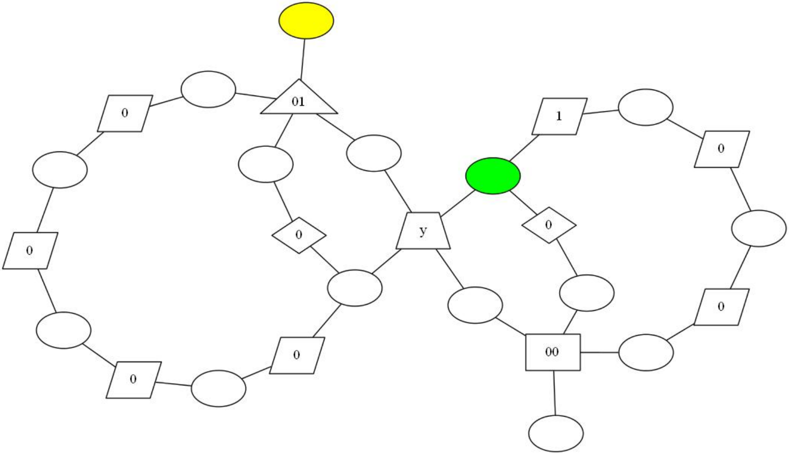

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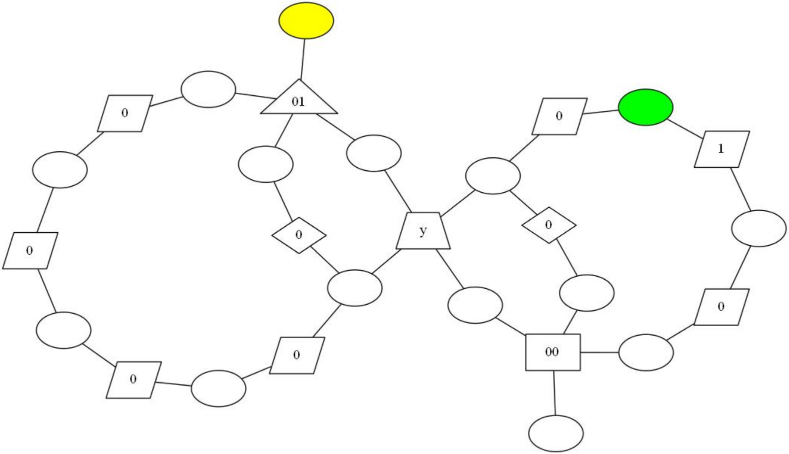

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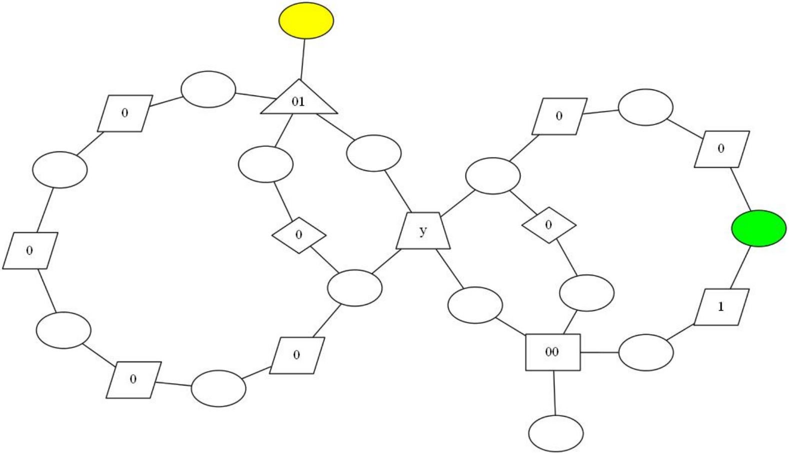

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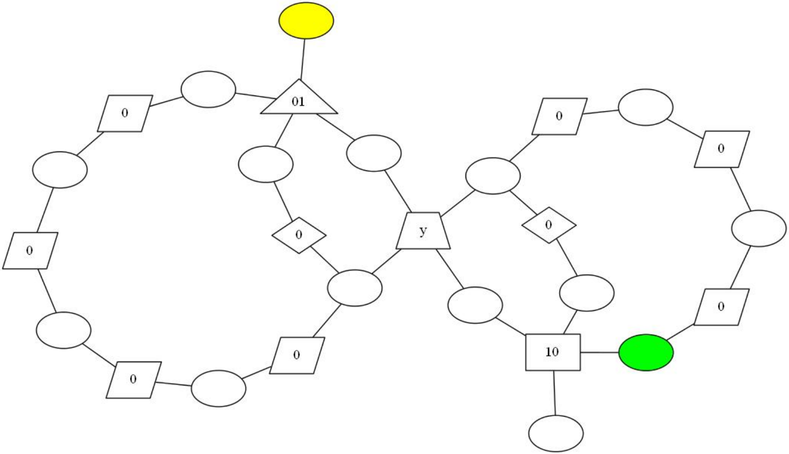

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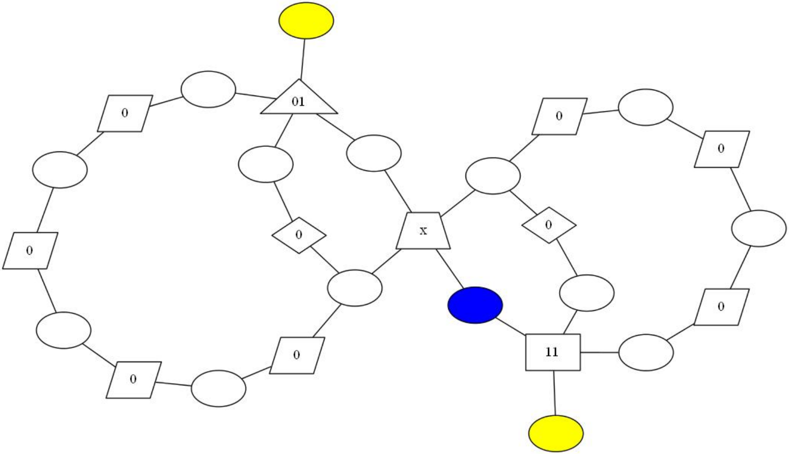

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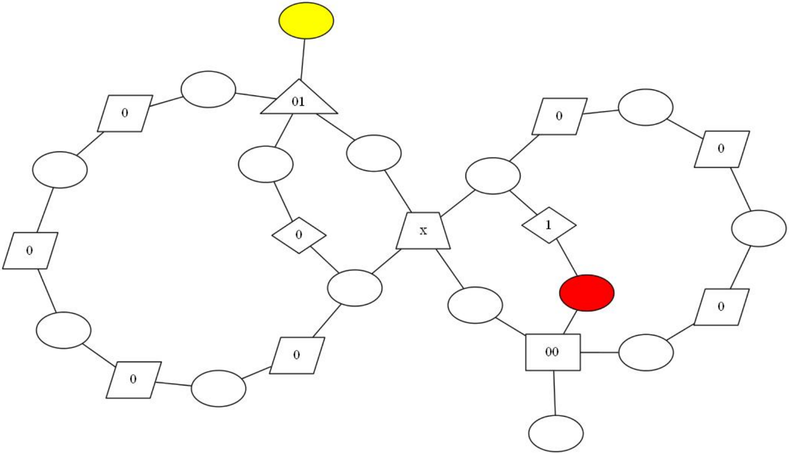

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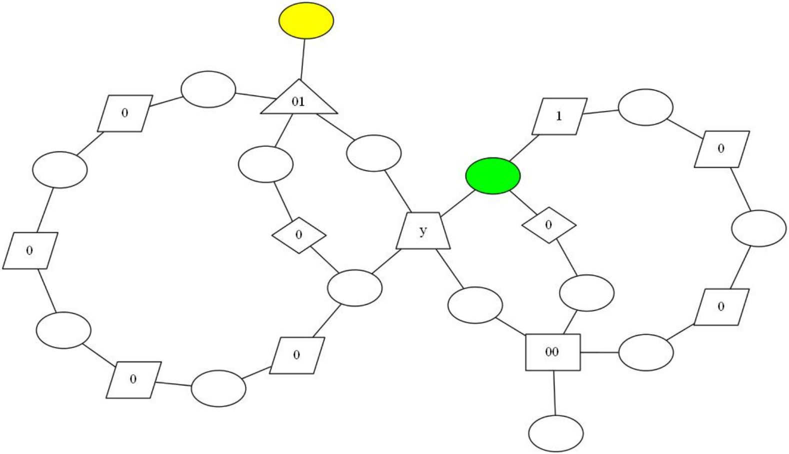

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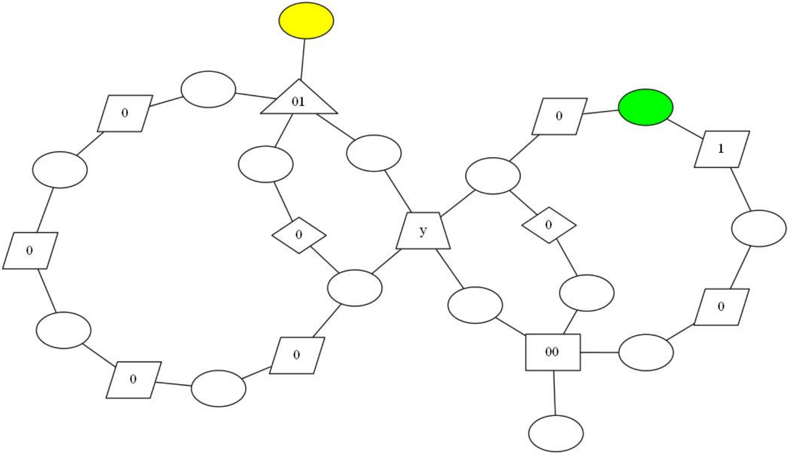

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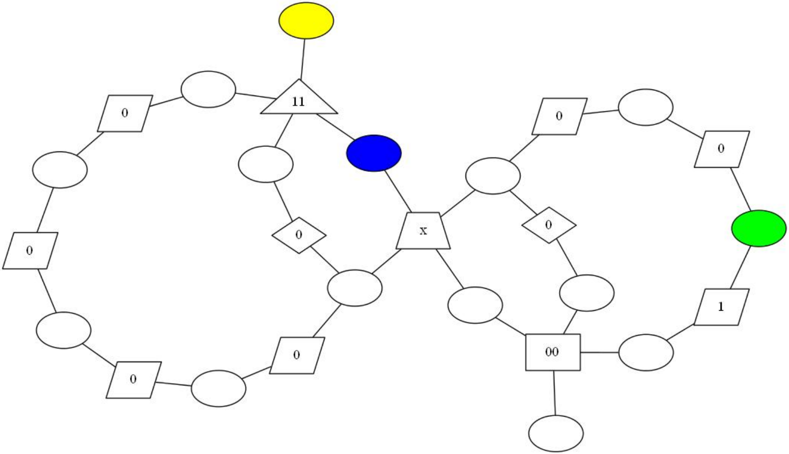

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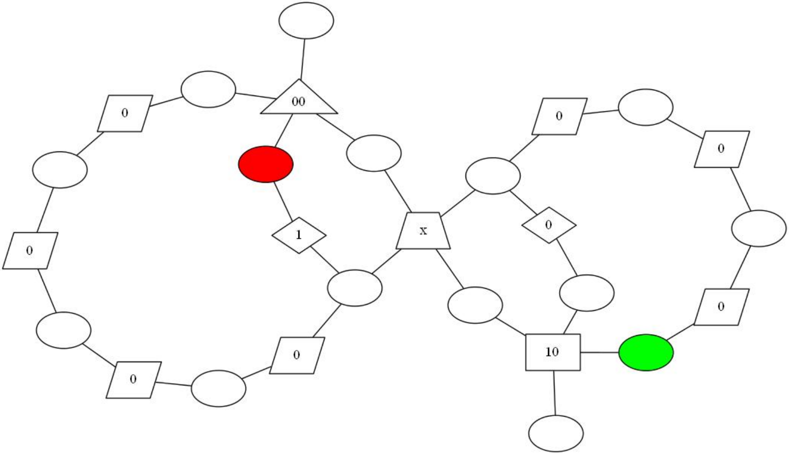

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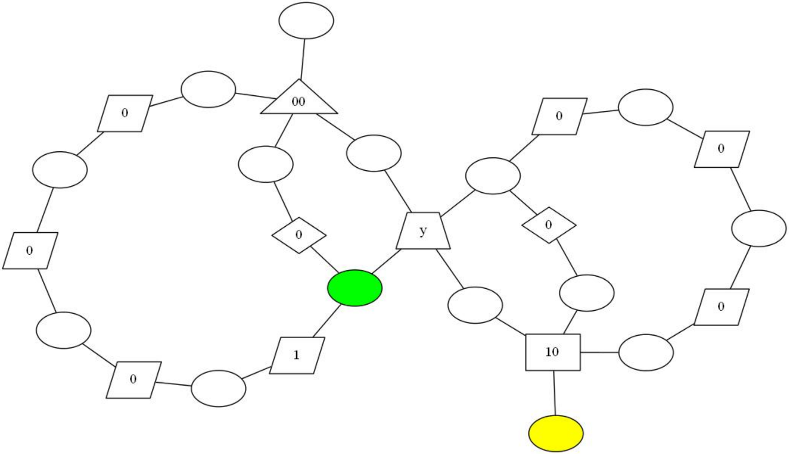

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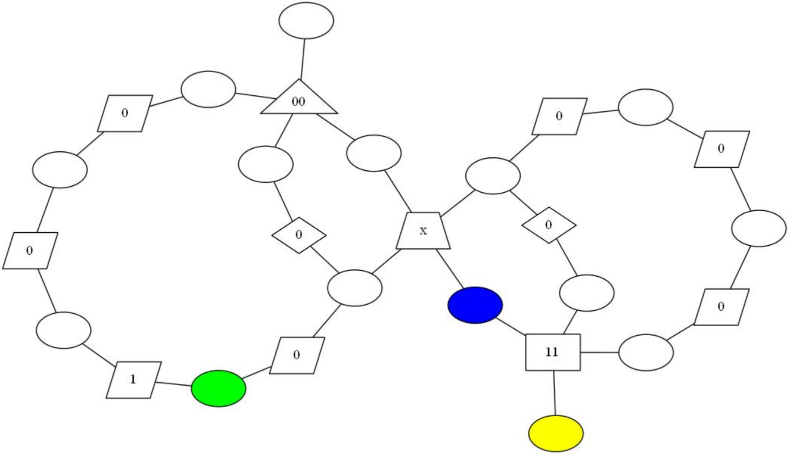

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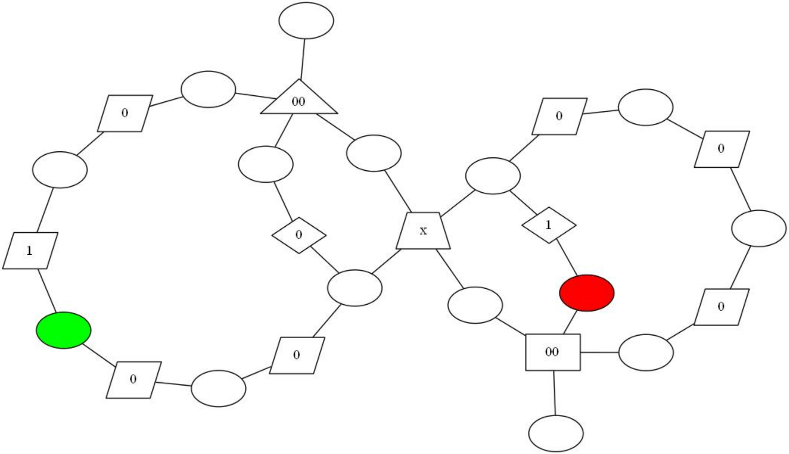

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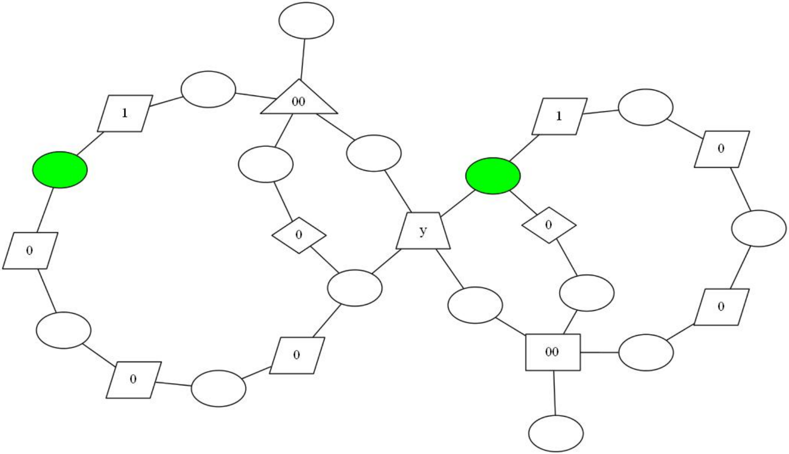

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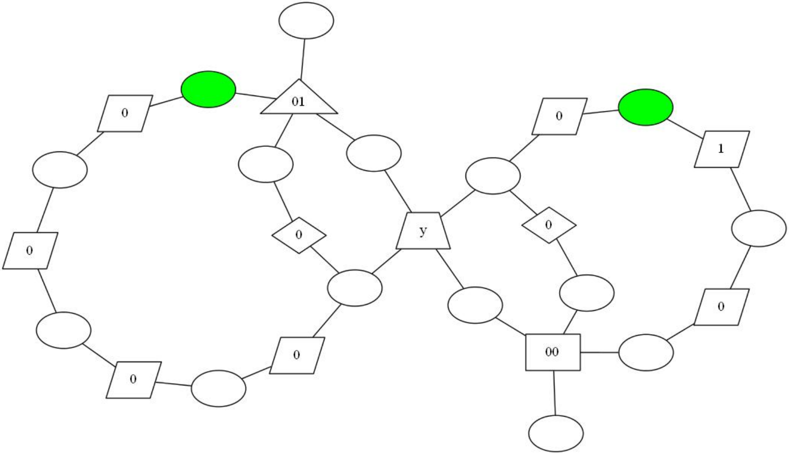

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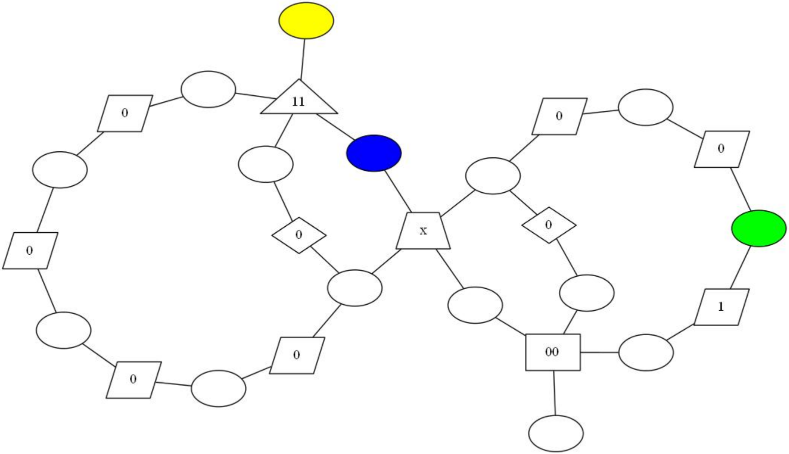

Summary

Networks of components have a compositional description in terms of the algebra of symmetric or braided monoidal categories in which each object has a commutative Frobenius algebra structure compatible with the tensor product. They also have a geometric description [11] - the free such algebra (in the symmetric case) is the category of cospans of monoidal graphs, arrows of which have a pictorial representation; in the braided case the geometry is more complicated, capturing not only the connection between components but also their entanglement.
These results are in the line introduced by Penrose [1], and Joyal and Street [3], and were obtained by Sabadini and Walters with collaborators Katis and Rosebrugh in earlier work, especially [6,7,8,9,11,12], beginning with the work on relations with Carboni [2] in 1987. The work has numerous antecedents - we mention just S. Eilenberg, S.L. Bloom, Z. Esik, Gh. Stefanescu. The algebra has connections with quantum field theory [10]).

The present work presents two developments. The first is some initial work in classifying tangled circuits; the second is a tool for composing cospans of graphs and calculating executions of nets of parallel automata.

Bibliography

- [1] R. Penrose, *Applications of negative dimensional tensors*, in Combinatorial mathematics and its applications, Academic Press (1971).
- [2] A. Carboni and R.F.C. Walters, *Cartesian bicategories I*, Journal of Pure and Applied Algebra, 49 (1987) 11-32.
- [3] A. Joyal, R.H. Street, *The geometry of tensor calculus I*, Adv. Math. 88 (1991) 55-112.
- [4] A. Joyal, R.H. Street, *Braided monoidal categories*, Adv. Math. 102 (1993) 20-87.
- [5] P. Katis, N. Sabadini, R.F.C. Walters, *Representing P/T nets in Span(Graph)*, Proceedings AMAST '97, SLNCS 1349 (1997) 307-321.

Bibliography

- [6] P. Katis, N. Sabadini, R.F.C. Walters, *Span(Graph): an algebra of transition systems*, Proceedings AMAST '97, SLNCS 1349 (1997) 322-336.
- [7] P. Katis, N. Sabadini, R.F.C. Walters, *A formalisation of the IWIM Model*, in: Proc. COORDINATION 2000,(Eds.) Porto A., Roman G.-C., LNCS 1906, Springer Verlag, (2000) 267-283.
- [8] P. Katis, N. Sabadini, R.F.C. Walters, *On the algebra of systems with feedback and boundary*, Rendiconti del Circolo Matematico di Palermo Serie II, Suppl. 63 (2000), 123-156.
- [9] P. Katis, N. Sabadini, R.F.C. Walters, *Compositional minimization in Span(Graph): Some examples*, Electronic Notes in Theoretical Computer Science 104C,(2004) 181-197.
- [10] J. Kock, *Frobenius algebras and 2D Topological Quantum Field Theories*, Cambridge University Press (2004).

Bibliography

- [11] R. Rosebrugh, N. Sabadini, R.F.C. Walters, *Generic commutative separable algebras and cospans of graphs*, Theory and Applications of Categories, 15, (2005) 264-277.
- [12] R. Rosebrugh, N. Sabadini, R.F.C. Walters, *Calculating colimits compositionally*, Montanari Festschrift, LNCS 5065,(2008) 581592
- [13] L. de Francesco Albasini, N. Sabadini and R.F.C. Walters, *An algebra of automata that includes both classical and quantum entities*, Electr. Notes Theor. Comput. Sc. 270 (2011) 263-270.
- [14] R. Rosebrugh, N. Sabadini, R.F.C. Walters, *Tangled Circuits*, arXiv:1110.0715 (2011).
- [15] Jennifer Lantair, Pawel Sobocinski, *WiCcA: LTS generation tool for wire calculus*, Proceeding CALCO'11 Proceedings of the 4th international conference on Algebra and coalgebra in computer science, (2011) 407-412.
- [16] Roberto Bruni, Hernan Melgratti, Ugo Montanari and Pawel Sobocinski, *Connector algebras for C/E and P/T nets*