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The Cantor frame

The existence of free frames and quotient frames allows the definition of a frame by generators and relations. Joyal [8] and Banaschewski [3] give a pointfree description of the real numbers in this way. The aim of this work is to give a pointfree description of the Cantor set. A topological characterization of it is given by Brouwer's Theorem [4]: The Cantor set is the unique totally disconnected, compact metric space with no isolated points. It can be shown (see, e.g. [1]) that the Cantor set is homeomorphic to the p-adic integers

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$$

for every prime number p. To give a pointfree description of the Cantor set, we specify the frame of $\mathbb{Z}_p$ by generators and relations. We use the fact that the open balls centered at integers generate the open subsets of $\mathbb{Z}_p$ and thus we think of them as the basic generators; we consider the (lattice) properties of these balls to determine the relations these elements must satisfy.

Set $|\mathbb{Z}| := \{p^{-n+1} \mid n \in \mathbb{N}\}$ and let $\mathcal{L}(\mathbb{Z}_p)$ be the frame generated by the elements $B_r(a)$, where $a \in \mathbb{Z}$ and $r \in |\mathbb{Z}|$, subject to the following relations:

(Q1) $B_s(b) \leq B_r(a)$ whenever $|a-b|_p < r$ and $s \leq r$.

(Q2) $B_r(a) \wedge B_s(b) = 0$ whenever $|a-b|_p \geq r$ and $s \leq r$.

(Q3) $1 = \bigvee \{B_r(a) : a \in \mathbb{Z},\ r \in |\mathbb{Z}|\}$.

(Q4) $B_r(a) = \bigvee \{B_s(b) : |a-b|_p < r,\ s < r,\ b \in \mathbb{Z},\ s \in |\mathbb{Z}|\}$.

We prove that $\mathcal{L}(\mathbb{Z}_p)$ is a spatial frame whose space of points is homeomorphic to $\mathbb{Z}_p$. In particular, we show with pointfree arguments that $\mathcal{L}(\mathbb{Q}_p)$ is 0-dimensional, (completely) regular, compact, and metrizable. Moreover, we show that the frame $\mathcal{L}(\mathbb{Z}_p)$ satisfies $cbd_{\mathcal{L}(\mathbb{Z}_p)}(0) = 0$, where $cbd_{\mathcal{L}(\mathbb{Z}_p)} : \mathcal{L}(\mathbb{Z}_p) \to \mathcal{L}(\mathbb{Z}_p)$, defined by

$$cbd_{\mathcal{L}(\mathbb{Z}_p)}(a) = \bigwedge \{x \in \mathcal{L}(\mathbb{Z}_p) \mid x \leq a \text{ and } (x \to a) = a\}$$

is the Cantor-Bendixson derivative (see, e.g. [10]). It follows that a frame L is isomorphic to $\mathcal{L}(\mathbb{Z}_p)$ if and only if L is a 0-dimensional compact regular metrizable frame with $cbd_L(0) = 0$. Finally, we give a point-free counterpart of the Hausdorff-Alexandroff Theorem which states that *every compact metric space is a continuous image of the Cantor space* (see, e.g. [2] and [6]). In point-free topology, we show that if L is a compact regular metrizable frame, then there is an injective frame homomorphism from L into $\mathcal{L}(\mathbb{Z}_2)$.

Joint work with Ángel Zaldívar.

References

[1] R. Alain M. *A Course in p-adic Analysis*. Springer, 2000.

[2] P. Alexandroff. Über stetige Abbildungen kompakter Räume. *Math. Ann.* 96(1):555–571, 1927.

[3] B. Banaschewski. The real numbers in pointfree topology, volume 12 of Textos de Matemática. Departamento de Matemática da Universidade de Coimbra, 1997.

[4] L. E. J. Brouwer. On the structure of perfect sets of points. *Proc. Koninklijke Akademie van Wetenschappen* 12:785–794, 1910.

[5] F. Q. Gouvêa. *p-adic Numbers: An Introduction*. Springer-Verlag, 1991.

[6] F. Hausdorff. *Mengenlehre, zweite, neubearbeitete Auflage*. Verlag Walter de Gruyter & Co., Berlin, 1927.

[7] K. Hensel. Über eine neue Begründung der Theorie der algebraischen Zahlen. *Journal für die reine und angewandte Mathematik*, 128:1–32, 1905.

[8] A. Joyal. Nouveaux fondaments de l'analyse. Lecture Notes, Montréal (Unpublished), 1973 and 1974.

[9] J. Picado and A. Pultr. *Frames and Locales: Topology without points*. Frontiers in Mathematics. Springer Basel, 2012.

[10] H. Simmons. Cantor-Bendixson properties of the assembly of a frame. In: *Leo Esakia on duality in modal and intuitionistic logics*, volume 4 of Outst. Contrib. Log., pages 217–255. Springer, Dordrecht, 2014.

[11] S. Willard. *General Topology*. Adison-Wesley Publishing Company, 1970.