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The frame of the metric hedgehog and a cardinal extension of normality

Topological hedgehogs keep generating interest in point-set topology as they are a rich source of counterexamples and applications.

In this talk we will present the hedgehog metric topology in a pointfree form, by specifying its generators and relations, and some of its main properties. We will also study collectionwise normality in frames, a concept originally introduced by A. Pultr ([4]) in connection with metrizability, and present the counterparts of Urysohn's separation and Tietze's extension theorems for continuous hedgehog-valued functions.

The frame of the metric hedgehog with κ spines

Let κ be some cardinal and let I be a set of cardinality κ. The *frame of the metric hedgehog with κ spines* is the frame $\mathfrak{L}(J(\kappa))$ presented by generators $(r,\text{—})_i$ and $(\text{—},r)$ for $r \in \mathbb{Q}$ and $i \in I$, subject to the defining relations

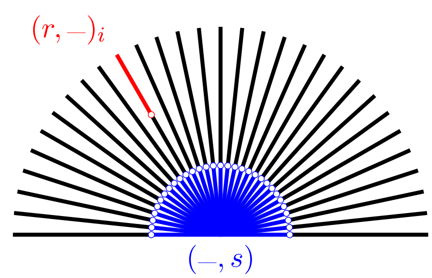
(h0) $(r,\text{—})_i \wedge (s,\text{—})_j = 0$ whenever $i \neq j$,

(h1) $(r,\text{—})_i \wedge (\text{—},s) = 0$ whenever $r \geq s$ and $i \in I$,

(h2) $\bigvee_{i\in I}(r_i,\text{—})_i \vee (\text{—},s) = 1$ whenever $r_i < s$ for all $i \in I$,

(h3) $(r,\text{—})_i = \bigvee_{s>r}(s,\text{—})_i$, for every $r \in \mathbb{Q}$ and $i \in I$,

(h4) $(\text{—},r) = \bigvee_{s<r}(\text{—},s)$, for every $r \in \mathbb{Q}$.



Note that $\mathfrak{L}(J(1))$ is precisely the frame of the extended reals $\mathfrak{L}(\overline{\mathbb{R}})$ and hence isomorphic to $\mathfrak{L}([0,1])$ (cf. [1, 2]).

We have the following:

(1) The spectrum $\Sigma\mathfrak{L}(J(\kappa))$ is homeomorphic to the classical metric hedgehog $J(\kappa)$.

(2) $\mathfrak{L}(J(\kappa))$ is a metric frame of weight $\kappa \cdot \aleph_0$ (cf. [4] and [3]).

(3) $\mathfrak{L}(J(\kappa))$ is complete in its metric uniformity (cf. [5]).

(4) The coproduct $\bigoplus_{n\in\mathbb{N}}\mathfrak{L}(J(\kappa))$ is universal in the class of metric frames of weight $\kappa \cdot \aleph_0$.

This presentation also allows us to define *continuous* (*metric*) *hedgehog-valued* functions on a frame L as frame homomorphisms $h\colon \mathfrak{L}(J(\kappa)) \to L$. Each continuous hedgehog-valued function determines a *join cozero κ-family*, i..e a disjoint collection $\{a_i\}_{i\in I}$, $|I| = \kappa$, of cozero elements of L such that $\bigvee_{i\in I} a_i$ is again a cozero element, and conversely.

Joint work with Imanol Mozo Carollo, Jorge Picado, and Joanne Walters-Wayland.

κ-collectionwise normality and the metric hedgehog

Being metrizable, the hedgehog frame is collectionwise normal [4, Theorem 2.5]. Recall that collectionwise normality is a stronger variant of normality introduced by A. Pultr in [4]: while a frame L is *normal* whether for any $x, y \in L$ satisfying $x \vee y = 1$ there exist disjoint $u, v \in L$ such that $x \vee u = 1 = y \vee v$, it is *collectionwise normal* if for each co-discrete system $\{x_i\}_{i \in I}$ there is a discrete $\{u_i\}_{i \in I}$ such that $x_i \vee u_i = 1$ for every $i \in I$. Here, by a *co-discrete* (resp. *discrete*) $\{x_i\}_{i \in I}$ it is meant a collection for which there is a cover C such that for each $c \in C$, $c \leq x_i$ (resp. $c \wedge x_i = 0$) for all i with possibly one exception. More generally, for a cardinal $\kappa \geq 2$, we say that L is *κ-collectionwise normal* if it satisfies the definition of collectionwise normality for sets I with cardinality $|I| \leq \kappa$. Hence collectionwise normality is κ-collectionwise normality for any cardinality κ. If $\kappa \leq \lambda$ are two cardinalities, then λ-collectionwise normality implies κ-collectionwise normality. Hence, κ-collectionwise normality implies normality for every κ.

We have the following:

(1) Any F_σ-sublocale (i.e. countable join of closed subocales) of a κ-collectionwise normal locale is κ-collectionwise normal.

(2) Let $h\colon M \to L$ be a one-to-one closed frame homomorphism. If L is κ-collectionwise normal, then so is M.

(3) **(Urysohn-type theorem)** A frame L is κ-collectionwise normal if and only if for each co-discrete system $\{x_i\}_{i \in I}$ with $|I| \leq \kappa$ there exists a continuous hedgehog-valued function $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $h((0, \_)_i^*) \leq x_i$ for each $i \in I$.

(4) **(Tietze-type theorem)** A frame L is κ-collectionwise normal if and only if for every closed sublocale $\mathfrak{c}(a)$ of L, each continuous hedgehog-valued function $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$ has an extension to L.

References

[1] B. Banaschewski, *The real numbers in Pointfree Topology*, Textos de Matemática, Vol. 12, University of Coimbra, 1997.

[2] B. Banaschewski, J. Gutiérrez García and J. Picado, Extended real functions in pointfree topology, *J. Pure Appl. Algebra* 216 (2012) 905–922.

[3] L. Español, J. Gutiérrez García and T. Kubiak, Separating families of locale maps and localic embeddings, *Algebra Univ.* 67 (2012) 105–112.

[4] A. Pultr, Remarks on metrizable locales, in: *Proc. of the 12th Winter School on Abstract Analysis*, pp. 247–258, *Rendiconti del Circolo Matematico di Palermo*, 1984.

[5] A. Pultr, Pointless uniformities. II: (Dia)metrization, *Comment. Math. Univ. Carolinae* 25 (1984) 105–120.