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Enriched topologies and the topological representation of semi-unital quantales

The topologization of the quantale of all closed right ideals of a non-commutative C^*-algebra is an old problem of non-commutative geometry. A first solution based on the identification of prime elements with $\mathbf{2}$-valued closed maps was given by F. Borceux and G. van den Bosche in 1989 (cf. [1]). In 2015, it has been shown that prime elements of right-sided and idempotent quantales are equivalent to three-valued strong quantale homomorphisms (cf. [4]). Hence it seems that enriched category theory, especially enriched topologies form an appropriate framework for the solution of the previous topologization problem.

Let Sup be the symmetric and monoidal closed category of complete lattices and join-preserving maps. Then unital quantales $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ are monoids in Sup.

As a first step we recall the fundamental fact that the $\mathfrak{Q}$-enriched version of Sup is isomorphic to the category $\mathsf{Mod}_r(\mathfrak{Q})$ of right $\mathfrak{Q}$-modules in Sup (cf. [6]). Then the $\mathfrak{Q}$-enriched power set of a given set X is the free right $\mathfrak{Q}$-module generated by X — i.e. the complete lattice $\mathfrak{Q}^X$ of all maps $X \xrightarrow{f} \mathfrak{Q}$ provided with the right action (cf. [5])

$$(f \boxdot \alpha)(x) = f(x) * \alpha, \qquad x \in X,\, \alpha \in \mathfrak{Q}.$$

Hence a $\mathfrak{Q}$-*enriched topology* on X is a right $\mathfrak{Q}$-submodule $\mathcal{T}$ of $\mathfrak{Q}^X$ provided with the additional properties:

— $\underline{\top} \in \mathcal{T}$,

— $f, g \in \mathcal{T} \quad \Longrightarrow \quad f * g \in \mathcal{T}$,

where $\underline{\top}$ is the constant map determined by the universal upper bound $\top$ in $\mathfrak{Q}$ and $(f * g)(x) = f(x) * g(x)$ for all $x \in X$.

A $\mathfrak{Q}$-enriched topological space $(X, \mathcal{T})$ is called *sober* if $(X, \mathcal{T})$ is a T_0-*space* — i.e. if $x_1 \neq x_2$ in X, then there exists $f \in \mathcal{T}$ such that $f(x_1) \neq f(x_2)$ — and every right $\mathfrak{Q}$-algebra morphism $\mathcal{T} \xrightarrow{h} \mathfrak{Q}$ is induced by an appropriate element $x_h \in X$ — i.e. $h(f) = f(x_h)$ for all $f \in \mathcal{T}$.

Let $\mathfrak{Q}_{\mathbf{2}}$ be the quantization of $\mathbf{2} = \{0, 1\}$ — this is the tensor product of the non-commutative, idempotent, left-sided three-chain C_3^ℓ with the non-commutative, idempotent, right-sided three-chain C_3^r (cf. [2, Sect. 2.5]).

On this basis the following is proved:

(1) Every spatial semi-unital quantale is equivalent to a six-valued — i.e. $\mathfrak{Q}_{\mathbf{2}}$-valued — topological space (cf. [3]).

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(2) If $\widehat{\mathfrak{Q}_{\mathbf{2}}}$ is the unitalization of $\mathfrak{Q}_{\mathbf{2}}$, then every $\mathfrak{Q}_{\mathbf{2}}$-valued topology generates a $\widehat{\mathfrak{Q}_{\mathbf{2}}}$-enriched topology.

As a corollary of (1) and (2) we obtain that every spatial semi-unital quantale is equivalent to an appropriate $\widehat{\mathfrak{Q}_{\mathbf{2}}}$-enriched sober space $(X, \mathcal{T})$, where X is the spectrum of some right $\widehat{\mathfrak{Q}_{\mathbf{2}}}$-algebra $\mathbb{L}$ in Sup — i.e. the set of all right $\widehat{\mathfrak{Q}_{\mathbf{2}}}$-algebra morphisms $\mathbb{L} \xrightarrow{k} \widehat{\mathfrak{Q}_{\mathbf{2}}}$ — and elements $\mathbb{A}_\alpha$ of $\mathcal{T}$ have the form:

$$\mathbb{A}_\alpha(k) = k(\alpha), \qquad k \in X,\, \alpha \in \mathbb{L}.$$

References

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