Order theory, enriched

Dirk Hofmann

CIDMA, Department of Mathematics, University of Aveiro, Portugal
dirk@ua.pt, http://sweet.ua.pt/dirk/

September 29, 2018

Happy Birthday, Aleš

Motivation and background

Motivation

The Name of the Rose

> *I started to write in March of 1978, moved by a vague idea: I wanted to poison a monk.* [a]

[a] Umberto Eco, Postille a ‚Il nome della rosa'.

Motivation

The Name of the Rose

I started to write in March of 1978, moved by a vague idea: I wanted to poison a monk. [a]

[a] Umberto Eco, Postille a ‚Il nome della rosa'.

Our motivation is less dramatic. . .

The kinds of structures which actually arise in the practice of geometry and analysis are far from being 'arbitrary' . . . , as concentrated in the thesis that fundamental structures are themselves categories. [a]

[a] F. William Lawvere. "Metric spaces, generalized logic, and closed categories". In: *Rendiconti del Seminario Matemàtico e Fisico di Milano* **43**.(1) (1973), pp. 135–166.

Lawvere's metric spaces

Metric space = category

enriched in $[0,\infty]$: a "hom-function" $a\colon X \times X \to [0,\infty]$ with

$$0 \geq a(x,x) \qquad \text{and} \qquad a(x,y) + a(y,z) \geq a(x,z).$$

Lawvere's metric spaces

Metric space = category

enriched in $[0, \infty]$: a "hom-function" $a\colon X \times X \to [0, \infty]$ with

$$0 \geq a(x, x) \qquad \text{and} \qquad a(x, y) + a(y, z) \geq a(x, z).$$

Relations and functions

Lawvere's metric spaces

Metric space = category

enriched in $[0, \infty]$: a "hom-function" $a \colon X \times X \to [0, \infty]$ with

$$0 \geq a(x,x) \qquad \text{and} \qquad a(x,y) + a(y,z) \geq a(x,z).$$

Relations and functions

- For every function $f \colon X \to Y$, $f \dashv f^{\circ}$ in **Rel**.

Lawvere's metric spaces

Metric space = category

enriched in $[0, \infty]$: a "hom-function" $a\colon X \times X \to [0, \infty]$ with

$$0 \geq a(x,x) \qquad \text{and} \qquad a(x,y) + a(y,z) \geq a(x,z).$$

Relations and functions

- For every function $f\colon X \to Y$, $f \dashv f^\circ$ in **Rel**.
- A relation $r\colon X \nrightarrow Y$ is a function $\iff$ $r \dashv s$ in **Rel**.

Lawvere's metric spaces

Metric space = category

enriched in $[0,\infty]$: a "hom-function" $a\colon X \times X \to [0,\infty]$ with

$$0 \geq a(x,x) \qquad \text{and} \qquad a(x,y) + a(y,z) \geq a(x,z).$$

Relations and functions

- For every function $f\colon X \to Y$, $f \dashv f^\circ$ in **Rel**.
- A relation $r\colon X \nrightarrow Y$ is a function $\iff$ $r \dashv s$ in **Rel**.

. . . and the ordered version

- Distributor (or order ideal . . .): relation $r\colon X \nrightarrow Y$ so that

$$\begin{aligned}(x \leq x') \,\&\, (x' \, r \, y) &\implies (x \, r \, y),\\ (x \, r \, y') \,\&\, (y' \leq y) &\implies (x \, r \, y).\end{aligned}$$

Lawvere's metric spaces

Metric space = category

enriched in $[0,\infty]$: a "hom-function" $a\colon X \times X \to [0,\infty]$ with

$$0 \geq a(x,x) \qquad \text{and} \qquad a(x,y) + a(y,z) \geq a(x,z).$$

Relations and functions

- For every function $f\colon X \to Y$, $f \dashv f^\circ$ in **Rel**.
- A relation $r\colon X \nrightarrow Y$ is a function $\iff$ $r \dashv s$ in **Rel**.

. . . and the ordered version

- Distributor (or order ideal . . .): relation $r\colon X \nrightarrow Y$ so that

$$\begin{aligned}(x \leq x') \,\&\, (x' \, r \, y) &\implies (x \, r \, y),\\ (x \, r \, y') \,\&\, (y' \leq y) &\implies (x \, r \, y).\end{aligned}$$

- A monotone map $f\colon X \to Y$ defines distributors $f_* \dashv f^*$:

$$x \, f_* \, y \quad \text{if} \quad f(x) \leq y, \qquad y \, f^* \, x \quad \text{if} \quad y \leq f(x).$$

Lawvere's metric spaces

Metric space = category

enriched in $[0,\infty]$: a "hom-function" $a\colon X\times X\to[0,\infty]$ with

$$0\geq a(x,x)\qquad\text{and}\qquad a(x,y)+a(y,z)\geq a(x,z).$$

Relations and functions

- For every function $f\colon X\to Y$, $f\dashv f^\circ$ in **Rel**.
- A relation $r\colon X\mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} Y$ is a function $\iff$ $r\dashv s$ in **Rel**.

... and the ordered version

- Distributor (or order ideal ...): relation $r\colon X\mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} Y$ so that

$$\begin{aligned}(x\leq x')\ \&\ (x'\,r\,y)&\implies(x\,r\,y),\\(x\,r\,y')\ \&\ (y'\leq y)&\implies(x\,r\,y).\end{aligned}$$

- A monotone map $f\colon X\to Y$ defines distributors $f_*\dashv f^*$:

$$x\,f_*\,y\quad\text{if}\quad f(x)\leq y,\qquad y\,f^*\,x\quad\text{if}\quad y\leq f(x).$$

- $r=f_*\iff r$ is left adjoint.

Lawvere's metric spaces

Metric space = category

enriched in $[0, \infty]$: a "hom-function" $a\colon X \times X \to [0, \infty]$ with

$$0 \geq a(x,x) \qquad \text{and} \qquad a(x,y) + a(y,z) \geq a(x,z).$$

Metric distributors...

... are $[0,\infty]$-enriched relations $\varphi\colon X \times Y \to [0,\infty]$ so that

$$a(x,x') + \varphi(x',y) \geq \varphi(x,y), \quad \varphi(x,y') + b(y',y) \geq \varphi(x,y).$$

Lawvere's metric spaces

Metric space = category

enriched in $[0,\infty]$: a "hom-function" $a\colon X \times X \to [0,\infty]$ with

$$0 \geq a(x,x) \qquad \text{and} \qquad a(x,y) + a(y,z) \geq a(x,z).$$

Metric distributors...

... are $[0,\infty]$-enriched relations $\varphi\colon X \times Y \to [0,\infty]$ so that

$$a(x,x') + \varphi(x',y) \geq \varphi(x,y), \quad \varphi(x,y') + b(y',y) \geq \varphi(x,y).$$

A metric map $f\colon X \to Y$ defines distributors $f_* \dashv f^*$:

$$f_*(x,y) = b(f(x),y), \qquad f^*(y,x) = f(y,f(x)).$$

Lawvere's metric spaces

Metric space = category

enriched in $[0,\infty]$: a "hom-function" $a\colon X \times X \to [0,\infty]$ with

$$0 \geq a(x,x) \qquad \text{and} \qquad a(x,y) + a(y,z) \geq a(x,z).$$

Metric distributors...

... are $[0,\infty]$-enriched relations $\varphi\colon X \times Y \to [0,\infty]$ so that

$$a(x,x') + \varphi(x',y) \geq \varphi(x,y), \quad \varphi(x,y') + b(y',y) \geq \varphi(x,y).$$

A metric map $f\colon X \to Y$ defines distributors $f_* \dashv f^*$:

$$f_*(x,y) = b(f(x),y), \qquad f^*(y,x) = f(y,f(x)).$$

Theorem (Lawvere (1973))

Every left adjoint distributor $\varphi\colon X \,{\circ\!\!\rightarrow}\, Y$ comes from a metric map if and only if Y is Cauchy-complete.

Assymetric Cauchy completeness

Directed distributors

The notions of forward and backward Cauchy sequences

$$\forall \varepsilon > 0 \, \forall m \geq n \, \ldots \; a(x_m, x_n) < \varepsilon \text{ and } \ldots$$

generalise *eventually increasing* resp. *decreasing* sequences.

Marcello M. Bonsangue, Franck van Breugel, and Jan Rutten. "Generalized metric spaces: completion, topology, and powerdomains via the Yoneda embedding". In: *Theoretical Computer Science* **193**.(1-2) (1998), pp. 1–51.

Assymetric Cauchy completeness

Directed distributors

The notions of forward and backward Cauchy sequences

$$\forall \varepsilon > 0\, \forall m \geq n \,\ldots\, a(x_m, x_n) < \varepsilon \text{ and } \ldots$$

generalise *eventually increasing* resp. *decreasing* sequences.

Theorem

Forward-Cauchy nets in metric spaces correspond precisely to those $[0,\infty]$-distributors $\psi\colon X \nrightarrow 1$ (called flat) where

$$\psi\cdot - : [0,\infty]\text{-}\mathbf{Dist}(1,X) \longrightarrow [0,\infty],\ \varphi \longmapsto \psi\cdot\varphi$$

preserves finite meets.

Steven Vickers. "Localic completion of generalized metric spaces. I". In: *Theory and Applications of Categories* **14**.(15) (2005), pp. 328–356.

Assymetric Cauchy completeness

Directed distributors

The notions of forward and backward Cauchy sequences

$$\forall \varepsilon > 0 \, \forall m \geq n \ldots a(x_m, x_n) < \varepsilon \text{ and } \ldots$$

generalise *eventually increasing* resp. *decreasing* sequences.

Theorem

Forward-Cauchy nets in metric spaces correspond precisely to those $[0,\infty]$-distributors $\psi \colon X \nrightarrow 1$ (called flat) where

$$\psi \cdot - : [0,\infty]\text{-}\mathbf{Dist}(1, X) \longrightarrow [0,\infty], \; \varphi \longmapsto \psi \cdot \varphi$$

preserves finite meets.

Remark

A downset $B \subseteq X$ is directed iff $\mathrm{Up}(X) \longrightarrow 2$, $A \longmapsto [B \cap A \neq \varnothing]$ preserves finite meets.

Order from topology

A fundamental adjunction ...

...linking topology and order:

$$\begin{tikzcd} \textbf{Top} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \textbf{Ord} \arrow[l, bend left] \end{tikzcd}$$

Order from topology

A fundamental adjunction ...

...linking topology and order:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Ord} \arrow[l, bend left] \end{tikzcd}$$

- Topology $\longmapsto$ natural order defined by $x \leq y$ if $\dot{x} \to y$.
- Order $\longmapsto$ Alexandroff topology.

Order from topology

A fundamental adjunction ...

... linking topology and order:

$$\mathbf{Top} \underset{}{\overset{}{\rightleftarrows}} \mathbf{Ord} \quad (\top)$$

- Topology $\longmapsto$ natural order defined by $x \leq y$ if $\dot{x} \to y$.
- Order $\longmapsto$ Alexandroff topology.
- There is also the Scott topology...

Order from topology

A fundamental adjunction . . .

. . . linking topology and order:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Ord} \arrow[l, bend left] \end{tikzcd}$$

- Topology $\longmapsto$ natural order defined by $x \leq y$ if $\dot{x} \to y$.
- Order $\longmapsto$ Alexandroff topology.
- There is also the Scott topology. . .

- Continuous lattice $=$ injective topological space.

Order from topology

A fundamental adjunction ...

...linking topology and order:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Ord} \arrow[l, bend left] \end{tikzcd}$$

- Topology $\longmapsto$ natural order defined by $x \leq y$ if $\dot{x} \to y$.
- Order $\longmapsto$ Alexandroff topology.
- There is also the Scott topology...

- Continuous lattice $=$ injective topological space.

Metric version?

Windels (2001): "Solve[order/topology == quasi-metric/x, x]".

Order from topology

A fundamental adjunction ...

...linking topology and order:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Ord} \arrow[l, bend left] \end{tikzcd}$$

- Topology $\longmapsto$ natural order defined by $x \leq y$ if $\dot{x} \to y$.
- Order $\longmapsto$ Alexandroff topology.
- There is also the Scott topology...

- Continuous lattice = injective topological space.

Metric version?

Windels (2001): "Solve[order/topology == quasi-metric/x, x]".

Idea: Use approach spaces instead of topological spaces.

Order from topology

A fundamental adjunction . . .

. . . linking topology and order:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Ord} \arrow[l, bend left] \end{tikzcd}$$

- Topology $\longmapsto$ natural order defined by $x \leq y$ if $\dot{x} \to y$.
- Order $\longmapsto$ Alexandroff topology.
- There is also the Scott topology. . .
  Metric variants: in Windels (2000) and Li and Zhang (2018).
- Continuous lattice $=$ injective topological space.

Metric version?

Windels (2001): "Solve[order/topology $==$ quasi-metric/x, x]".

<u>Idea</u>: Use approach spaces instead of topological spaces.

Order from topology

A fundamental adjunction ...

...linking topology and order:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Ord} \arrow[l, bend left] \end{tikzcd}$$

- Topology $\longmapsto$ natural order defined by $x \leq y$ if $\dot{x} \to y$.
- Order $\longmapsto$ Alexandroff topology.
- There is also the Scott topology...
  Metric variants: in Windels (2000) and Li and Zhang (2018).
- Continuous lattice = injective topological space.
  Metric variant: Gutierres and Hofmann (2013).

Metric version?

Windels (2001): "Solve[order/topology == quasi-metric/x, x]".

<u>Idea</u>: Use approach spaces instead of topological spaces.

Ordered topological structures

Order and topology

Ordered topological structures

- Sets X equipped with order and topology so that the order relation is closed in $X \times X$.

  Leopoldo Nachbin. *Topologia e Ordem*. University of Chicago Press, 1950

Order and topology

Ordered topological structures

- Sets X equipped with order and topology so that the order relation is closed in $X \times X$.

  Leopoldo Nachbin. *Topologia e Ordem*. University of Chicago Press, 1950

- Of particular interest to us are ordered compact Hausdorff spaces, where we have:

$$\mathbf{PosComp} \sim \mathbf{StablyComp}.$$

A topological space is stably compact whenever X is sober, locally compact and stable.

Gerhard Gierz et al. *A compendium of continuous lattices.* Berlin: Springer-Verlag, 1980. xx + 371

Topology and order (via convergence)

Theorem

For a compact Hausdorff topology $\alpha\colon UX \to X$ and an order relation $\leq\colon X \nrightarrow X$, the following are equivalent:

(i) The order is closed in $X \times X$.

(ii) $\alpha\colon (UX, U{\leq}) \longrightarrow (X, \leq)$ is monotone.

In general, for a relation $r\colon X \nrightarrow Y$ one defines

$$\mathfrak{x}(Ur)\mathfrak{y} \qquad \textit{whenever} \qquad \forall A, B\,\exists x, y \,.\, x\,r\,y.$$

Walter Tholen. "Ordered topological structures". In: *Topology and its Applications* **156**.(12) (2009), pp. 2148–2157.

Topology and order (via convergence)

Theorem

For a compact Hausdorff topology $\alpha\colon UX \to X$ and an order relation $\leq\colon X \nrightarrow X$, the following are equivalent:

(i) The order is closed in $X \times X$.

(ii) $\alpha\colon (UX, U{\leq}) \longrightarrow (X, \leq)$ is monotone.

Theorem

The ultrafilter monad $\mathbb{U}$ on **Set** *extends to* **Ord** *and then*

$$\mathbf{Ord}^{\mathbb{U}} \sim \mathbf{OrdCH}.$$

Topology and order (via convergence)

Theorem

For a compact Hausdorff topology $\alpha\colon UX \to X$ and an order relation $\leq\colon X \nrightarrow X$, the following are equivalent:

(i) The order is closed in $X \times X$.

(ii) $\alpha\colon (UX, U{\leq}) \longrightarrow (X, \leq)$ is monotone.

Theorem

The ultrafilter monad $\mathbb{U}$ on **Set** *extends to* **Ord** *and then*

$$\mathbf{Ord}^{\mathbb{U}} \sim \mathbf{OrdCH}.$$

Remark

The canonical functor $\mathbf{Ord}^{\mathbb{U}} \longrightarrow$ **Top** with

$$(X, \leq, \alpha) \longmapsto (X, \leq \cdot \alpha : UX \to X \nrightarrow X)$$

restricts to an equivalence **PosComp** $\sim$ **StablyComp**.

Metric compact Hausdorff space

Extending the monad

The ultrafilter monad $\mathbb{U}$ on **Set** extends to **Met** via

$$Ud(\mathfrak{x}, \mathfrak{y}) = \bigvee_{A,B} \bigwedge_{x,y} d(x, y),$$

for a metric d on X.

Metric compact Hausdorff space

Extending the monad

The ultrafilter monad $\mathbb{U}$ on **Set** extends to **Met** via

$$Ud(\mathfrak{x}, \mathfrak{y}) = \bigvee_{A,B} \bigwedge_{x,y} d(x, y),$$

for a metric d on X.

Definition

A metric compact Hausdorff space is an algebra for $\mathbb{U}$ on **Met** (a compact Hausdorff space with a compatible metric).

Metric compact Hausdorff space

Extending the monad

The ultrafilter monad $\mathbb{U}$ on **Set** extends to **Met** via

$$Ud(\mathfrak{x}, \mathfrak{y}) = \bigvee_{A,B} \bigwedge_{x,y} d(x, y),$$

for a metric d on X.

Definition

A metric compact Hausdorff space is an algebra for $\mathbb{U}$ on **Met** (a compact Hausdorff space with a compatible metric).

Remark

Every compact metric space is a metric compact Hausdorff space.[a]

[a] Dirk Hofmann and Carla D. Reis. "Convergence and quantale-enriched categories". In: *Categories and General Algebraic Structures with Applications* **9**.(1) (2018), pp. 77–138.

Theorem

There is a canonical comparison functor

$$\mathbf{MetCH} \longrightarrow \mathbf{App}$$

sending (X, d, α) *to* $(X, d \cdot \alpha)$. $\quad (a(\mathfrak{x}, x) = d(\alpha(\mathfrak{x}), x))$

Theorem

There is a canonical comparison functor

$$\mathbf{MetCH} \longrightarrow \mathbf{App}$$

sending (X, d, α) *to* $(X, d \cdot \alpha)$. $\quad (a(\mathfrak{x}, x) = d(\alpha(\mathfrak{x}), x))$

Here, separated metric compact Hausdorff spaces correspond precisely to

- *core-compact (somehow exponentiable),*
- *sober (ask in two minutes) and*
- *stable (dont ask)*

approach spaces.

Vietoris functors

The lower Vietoris functor

- For a topological space X, the lower Vietoris space of X is
$$VX = \{A \subseteq X \mid A \text{ is closed}\}$$
with the topology generated by
$$\{A \in VX \mid A \cap B \neq \varnothing\} \qquad (B \text{ open}).$$
For a continuous map $f : X \to Y$, the map
$$Vf : VX \longrightarrow VY,\, A \longmapsto \overline{f(A)}$$
is continuous too.

Vietoris functors

The lower Vietoris functor

- For a topological space X, the lower Vietoris space of X is
$$VX = \{A \subseteq X \mid A \text{ is closed}\}$$
with the topology generated by
$$\{A \in VX \mid A \cap B \neq \varnothing\} \qquad (B \text{ open}).$$
For a continuous map $f : X \to Y$, the map
$$Vf : VX \longrightarrow VY,\ A \longmapsto \overline{f(A)}$$
is continuous too.
- This defines $V : \mathbf{Top} \to \mathbf{Top}$, which is part of a monad $\mathbb{V} = (V, m, e)$ on **Top** (almost as the powerset monad).

Vietoris functors

The lower Vietoris functor

- For a topological space X, the lower Vietoris space of X is
$$VX = \{A \subseteq X \mid A \text{ is closed}\}$$
with the topology generated by
$$\{A \in VX \mid A \cap B \neq \varnothing\} \qquad (B \text{ open}).$$
For a continuous map $f : X \to Y$, the map
$$Vf : VX \longrightarrow VY,\ A \longmapsto \overline{f(A)}$$
is continuous too.
- This defines $V : \mathbf{Top} \to \mathbf{Top}$, which is part of a monad $\mathbb{V} = (V, m, e)$ on **Top** (almost as the powerset monad).
- $\mathbb{V}$ restricts to a monad on **StablyComp** $\sim$ **PosComp**.

Vietoris functors

The lower Vietoris functor

- For a topological space X, the lower Vietoris space of X is
$$VX = \{A \subseteq X \mid A \text{ is closed}\}$$
with the topology generated by
$$\{A \in VX \mid A \cap B \neq \varnothing\} \qquad (B \text{ open}).$$
For a continuous map $f : X \to Y$, the map
$$Vf : VX \longrightarrow VY,\ A \longmapsto \overline{f(A)}$$
is continuous too.

Metric version

Vietoris functors

The lower Vietoris functor

- For a topological space X, the lower Vietoris space of X is
$$VX = \{A \subseteq X \mid A \text{ is closed}\}$$
with the topology generated by
$$\{A \in VX \mid A \cap B \neq \varnothing\} \qquad (B \text{ open}).$$
For a continuous map $f : X \to Y$, the map
$$Vf : VX \longrightarrow VY,\, A \longmapsto \overline{f(A)}$$
is continuous too.

Metric version

- topological space $\rightsquigarrow$ approach space,

Vietoris functors

The lower Vietoris functor

- For a topological space X, the lower Vietoris space of X is
$$VX = \{A \subseteq X \mid A \text{ is closed}\}$$
with the topology generated by
$$\{A \in VX \mid A \cap B \neq \varnothing\} \qquad (B \text{ open}).$$
For a continuous map $f : X \to Y$, the map
$$Vf : VX \longrightarrow VY,\, A \longmapsto \overline{f(A)}$$
is continuous too.

Metric version

- topological space $\rightsquigarrow$ approach space,
- $A \subseteq X$ closed $\rightsquigarrow$ $\varphi\colon X \to [0,\infty]$ approach map.

Vietoris functors

The lower Vietoris functor

- For a topological space X, the lower Vietoris space of X is
$$VX = \{A \subseteq X \mid A \text{ is closed}\}$$
with the topology generated by
$$\{A \in VX \mid A \cap B \neq \varnothing\} \qquad (B \text{ open}).$$
For a continuous map $f : X \to Y$, the map
$$Vf : VX \longrightarrow VY,\; A \longmapsto \overline{f(A)}$$
is continuous too.

Metric version

- topological space $\rightsquigarrow$ approach space,
- $A \subseteq X$ closed $\rightsquigarrow$ $\varphi\colon X \to [0, \infty]$ approach map.
- We obtain a monad on **App** which restricts to "stably compact approach spaces" (= metric compact Hausdorff spaces).

Duality theory

Frames

We all know the dual adjunction

$$\begin{tikzcd} \mathbf{Top}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Frm} \arrow[l, bend left] \end{tikzcd}$$

space $X \mapsto \mathcal{O}X \simeq \mathrm{hom}(X, 2)$, frame $L \mapsto \mathrm{spec}(L) \simeq \mathrm{hom}(L, 2)$.

Duality theory

Frames

We all know the dual adjunction

$$\begin{tikzcd} \mathbf{Top}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Frm} \arrow[l, bend left] \end{tikzcd}$$

space $X \mapsto \mathcal{O}X \simeq \mathrm{hom}(X, 2)$, frame $L \mapsto \mathrm{spec}(L) \simeq \mathrm{hom}(L, 2)$.

Approach frame

Banaschewski, Lowen, and Van Olmen (2006):

$$\begin{tikzcd} \mathbf{App}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{AFrm} \arrow[l, bend left] \end{tikzcd}$$

Duality theory

Frames

We all know the dual adjunction

$$
\begin{tikzcd}
\mathbf{Top}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Frm} \arrow[l, bend left]
\end{tikzcd}
$$

space $X \mapsto \mathcal{O}X \simeq \mathrm{hom}(X, 2)$, frame $L \mapsto \mathrm{spec}(L) \simeq \mathrm{hom}(L, 2)$.

Approach frame

Banaschewski, Lowen, and Van Olmen (2006):

$$
\begin{tikzcd}
\mathbf{App}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{AFrm} \arrow[l, bend left]
\end{tikzcd}
$$

- approach frame = (co)frame with actions of $[0, \infty]$,

Duality theory

Frames

We all know the dual adjunction

$$\begin{tikzcd} \mathbf{Top}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Frm} \arrow[l, bend left] \end{tikzcd}$$

space $X \mapsto \mathcal{O}X \simeq \hom(X, 2)$, frame $L \mapsto \operatorname{spec}(L) \simeq \hom(L, 2)$.

Approach frame

Banaschewski, Lowen, and Van Olmen (2006):

$$\begin{tikzcd} \mathbf{App}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{AFrm} \arrow[l, bend left] \end{tikzcd}$$

- approach frame = (co)frame with actions of $[0, \infty]$,

Alternatively, a metric space with all weighted limits and finite weigthed colimits.

Frames

We all know the dual adjunction

$$\begin{tikzcd} \mathbf{Top}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Frm} \arrow[l, bend left] \end{tikzcd}$$

space $X \mapsto \mathcal{O}X \simeq \mathrm{hom}(X, 2)$, frame $L \mapsto \mathrm{spec}(L) \simeq \mathrm{hom}(L, 2)$.

Approach frame

Banaschewski, Lowen, and Van Olmen (2006):

$$\begin{tikzcd} \mathbf{App}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{AFrm} \arrow[l, bend left] \end{tikzcd}$$

- approach frame = (co)frame with actions of $[0, \infty]$,
- space $X \longmapsto \mathcal{O}X \simeq \mathrm{hom}(X, [0, \infty])$,

Duality theory

Frames

We all know the dual adjunction

$$\begin{tikzcd} \mathbf{Top}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Frm} \arrow[l, bend left] \end{tikzcd}$$

space $X \mapsto \mathcal{O}X \simeq \mathrm{hom}(X, 2)$, frame $L \mapsto \mathrm{spec}(L) \simeq \mathrm{hom}(L, 2)$.

Approach frame

Banaschewski, Lowen, and Van Olmen (2006):

$$\begin{tikzcd} \mathbf{App}^{\mathrm{op}} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{AFrm} \arrow[l, bend left] \end{tikzcd}$$

- approach frame = (co)frame with actions of $[0, \infty]$,
- space $X \longmapsto \mathcal{O}X \simeq \mathrm{hom}(X, [0, \infty])$,
- frame $L \longmapsto \mathrm{spec}(L) \simeq \mathrm{hom}(L, [0, \infty])$.

Our motivation: Stone (and Halmos) dualities

Stone-Halmos dualities

Stone (1936), Stone (1938), Priestley (1970), and Priestley (1972)

$$\mathbf{Spec} \sim \mathbf{Priest} \underset{\hom(-,2)}{\overset{\hom(-,2)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathbf{DL}^{\mathrm{op}}, \qquad \mathbf{BooSp} \underset{\hom(-,2)}{\overset{\hom(-,2)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathbf{BA}^{\mathrm{op}}.$$

- Priestley space = partially ordered compact space X so that $(X \to 2)$ is point-separating and initial.
- Spectral space = stably compact space X so that $(X \to 2)$ is point-separating and initial.

Stone-Halmos dualities

Stone (1936), Stone (1938), Priestley (1970), and Priestley (1972)

$$\mathbf{Spec} \sim \mathbf{Priest} \underset{\text{hom}(-,2)}{\overset{\text{hom}(-,2)}{\rightleftarrows}} \mathbf{DL}^{\mathrm{op}}, \qquad \mathbf{BooSp} \underset{\text{hom}(-,2)}{\overset{\text{hom}(-,2)}{\rightleftarrows}} \mathbf{BA}^{\mathrm{op}}.$$

- Priestley space = partially ordered compact space X so that $(X \to 2)$ is point-separating and initial.
- Spectral space = stably compact space X so that $(X \to 2)$ is point-separating and initial.

Halmos (1956) and Cignoli, Lafalce, and Petrovich (1991)

$$\mathbf{Priest}_{\mathbb{V}} \sim \mathbf{DL}^{\mathrm{op}}_{\perp,\vee} \qquad \text{and} \qquad \mathbf{BooSp}_{\mathbb{V}} \sim \mathbf{BA}^{\mathrm{op}}_{\perp,\vee}.$$

Stone-Halmos dualities

Stone (1936), Stone (1938), Priestley (1970), and Priestley (1972)

$$\mathbf{Spec} \sim \mathbf{Priest} \underset{\mathrm{hom}(-,2)}{\overset{\mathrm{hom}(-,2)}{\rightleftarrows}} \mathbf{DL}^{\mathrm{op}}, \qquad \mathbf{BooSp} \underset{\mathrm{hom}(-,2)}{\overset{\mathrm{hom}(-,2)}{\rightleftarrows}} \mathbf{BA}^{\mathrm{op}}.$$

- Priestley space = partially ordered compact space X so that $(X \to 2)$ is point-separating and initial.
- Spectral space = stably compact space X so that $(X \to 2)$ is point-separating and initial.

Halmos (1956) and Cignoli, Lafalce, and Petrovich (1991)

$$\mathbf{Priest}_{\mathbb{V}} \sim \mathbf{DL}^{\mathrm{op}}_{\perp,\vee} \qquad \text{and} \qquad \mathbf{BooSp}_{\mathbb{V}} \sim \mathbf{BA}^{\mathrm{op}}_{\perp,\vee}.$$

Our aim

$$\mathbf{PosComp}_{\mathbb{V}} \xrightarrow{\mathrm{hom}(-,[0,\infty])} ???^{\mathrm{op}}$$

Stone-Halmos dualities

Stone (1936), Stone (1938), Priestley (1970), and Priestley (1972)

$$\mathbf{Spec} \sim \mathbf{Priest} \underset{\mathrm{hom}(-,2)}{\overset{\mathrm{hom}(-,2)}{\rightleftarrows}} \mathbf{DL}^{\mathrm{op}}, \qquad \mathbf{BooSp} \underset{\mathrm{hom}(-,2)}{\overset{\mathrm{hom}(-,2)}{\rightleftarrows}} \mathbf{BA}^{\mathrm{op}}.$$

- Priestley space = partially ordered compact space X so that $(X \to 2)$ is point-separating and initial.
- Spectral space = stably compact space X so that $(X \to 2)$ is point-separating and initial.

Halmos (1956) and Cignoli, Lafalce, and Petrovich (1991)

$$\mathbf{Priest}_{\mathbb{V}} \sim \mathbf{DL}^{\mathrm{op}}_{\perp,\vee} \qquad \text{and} \qquad \mathbf{BooSp}_{\mathbb{V}} \sim \mathbf{BA}^{\mathrm{op}}_{\perp,\vee}.$$

Our aim

$$\mathbf{PosComp}_{\mathbb{V}} \xrightarrow{\mathrm{hom}(-,[0,\infty])} \{\text{metric distributive lattice}\}^{\mathrm{op}}$$

Stone-Halmos dualities

Our aim

$$\mathbf{PosComp}_{\mathbb{V}} \xrightarrow{\hom(-,[0,\infty])} \{\text{metric distributive lattice}\}^{\mathrm{op}}$$

Related work

- Gelfand (1941): $\mathbf{CompHaus} \sim C^*\text{-}\mathbf{Alg}^{\mathrm{op}}$.

Stone-Halmos dualities

Our aim

$$\mathbf{PosComp}_{\mathbb{V}} \xrightarrow{\mathsf{hom}(-,[0,\infty])} \{\text{metric distributive lattice}\}^{\mathrm{op}}$$

Related work

- Gelfand (1941): $\mathbf{CompHaus} \sim C^{*}\text{-}\mathbf{Alg}^{\mathrm{op}}$.
- Jung, Kegelmann, and Moshier (2001):

$$\mathbf{StablyComp}_{\mathbb{V}} \sim \mathbf{StContDLat}^{\mathrm{op}}_{\bigvee,\ll}.$$

Stone-Halmos dualities

Our aim

$$\mathbf{PosComp}_{\mathbb{V}} \xrightarrow{\mathsf{hom}(-,[0,\infty])} \{\text{metric distributive lattice}\}^{\mathrm{op}}$$

Related work

- Gelfand (1941): $\mathbf{CompHaus} \sim C^*\text{-}\mathbf{Alg}^{\mathrm{op}}$.
- Jung, Kegelmann, and Moshier (2001):

$$\mathbf{StablyComp}_{\mathbb{V}} \sim \mathbf{StContDLat}^{\mathrm{op}}_{\bigvee,\ll}.$$

Note: From this we can deduce that $\mathbf{StablyComp}_{\mathbb{V}}$ is idempotent split complete.

Stone-Halmos dualities

Our aim

$$\mathbf{PosComp}_{\mathbb{V}} \xrightarrow{\hom(-,[0,\infty])} \{\text{metric distributive lattice}\}^{\mathrm{op}}$$

Related work

- Gelfand (1941): $\mathbf{CompHaus} \sim C^*\text{-}\mathbf{Alg}^{\mathrm{op}}$.
- Jung, Kegelmann, and Moshier (2001):

$$\mathbf{StablyComp}_{\mathbb{V}} \sim \mathbf{StContDLat}^{\mathrm{op}}_{\bigvee,\ll}.$$

<u>Note</u>: From this we can deduce that $\mathbf{StablyComp}_{\mathbb{V}}$ is idempotent split complete.

Metric distributive lattice

= metric space which is "finitaly cocomplete" and has a commutative monoid structure which preserves finite colimits in each variable.

Some facts about (partially) ordered compact spaces

Theorem

- **OrdCH** *is complete and cocomplete. Moreover, the full subcategory* **PosComp** $\hookrightarrow$ **OrdCH** *is reflective.*

Some facts about (partially) ordered compact spaces

Theorem

- **OrdCH** *is complete and cocomplete. Moreover, the full subcategory* **PosComp** $\hookrightarrow$ **OrdCH** *is reflective.*
- *The unit interval* $[0, 1]$ *is is an initial cogenerator on* **PosComp**.[a]

[a] Leopoldo Nachbin. *Topologia e Ordem*. University of Chicago Press, 1950.

Some facts about (partially) ordered compact spaces

Theorem

- **OrdCH** *is complete and cocomplete. Moreover, the full subcategory* **PosComp** $\hookrightarrow$ **OrdCH** *is reflective.*
- *The unit interval* $[0,1]$ *is is an initial cogenerator on* **PosComp**.
- *The unit interval* $[0,1]$ *is injective in* **PosComp** *with respect to embeddings.*[a]

[a] Leopoldo Nachbin. *Topologia e Ordem*. University of Chicago Press, 1950.

Some facts about (partially) ordered compact spaces

Theorem

- **OrdCH** *is complete and cocomplete. Moreover, the full subcategory* **PosComp** $\hookrightarrow$ **OrdCH** *is reflective.*
- *The unit interval* $[0, 1]$ *is is an initial cogenerator on* **PosComp**.
- *The unit interval* $[0, 1]$ *is injective in* **PosComp** *with respect to embeddings.*
- *The regular monomorphisms in* **PosComp** *are the embeddings, and the epimorphisms are the surjections.*

Some facts about (partially) ordered compact spaces

Theorem

- **OrdCH** *is complete and cocomplete. Moreover, the full subcategory* **PosComp** $\hookrightarrow$ **OrdCH** *is reflective.*
- *The unit interval* $[0, 1]$ *is is an initial cogenerator on* **PosComp**.
- *The unit interval* $[0, 1]$ *is injective in* **PosComp** *with respect to embeddings.*
- *The regular monomorphisms in* **PosComp** *are the embeddings, and the epimorphisms are the surjections.*
- $\mathbf{PosComp}^{\mathrm{op}}$ *is a quasivariety.*

  Recall: A category is equivalent to a quasivariety iff it has a regularly projective regular generator with copowers and coequalizers of pseudoequivalences.

Some facts about (partially) ordered compact spaces

Theorem

- **OrdCH** *is complete and cocomplete. Moreover, the full subcategory* **PosComp** $\hookrightarrow$ **OrdCH** *is reflective.*
- *The unit interval* $[0,1]$ *is is an initial cogenerator on* **PosComp**.
- *The unit interval* $[0,1]$ *is injective in* **PosComp** *with respect to embeddings.*
- *The regular monomorphisms in* **PosComp** *are the embeddings, and the epimorphisms are the surjections.*
- $\mathbf{PosComp}^{\mathrm{op}}$ *is a quasivariety.*
- *The* $\aleph_1$*-copresentable objects in* **PosComp** *are precisely the metrisable partially ordered compact spaces.*[a]

[a] Dirk Hofmann, Renato Neves, and Pedro Nora. "Generating the algebraic theory of $C(X)$: the case of partially ordered compact spaces". In: *Theory and Applications of Categories* **33**.(12) (2018), pp. 276–295.

Some facts about (partially) ordered compact spaces

Theorem

- **OrdCH** *is complete and cocomplete. Moreover, the full subcategory* **PosComp** $\hookrightarrow$ **OrdCH** *is reflective.*
- *The unit interval* $[0,1]$ *is is an initial cogenerator on* **PosComp**.
- *The unit interval* $[0,1]$ *is injective in* **PosComp** *with respect to embeddings.*
- *The regular monomorphisms in* **PosComp** *are the embeddings, and the epimorphisms are the surjections.*
- $\mathbf{PosComp}^{\mathrm{op}}$ *is a quasivariety.*
- *The* $\aleph_1$*-copresentable objects in* **PosComp** *are precisely the metrisable partially ordered compact spaces. In particular,* $[0,1]$ *is* $\aleph_1$*-copresentable.*

Some facts about (partially) ordered compact spaces

Theorem

- **OrdCH** *is complete and cocomplete. Moreover, the full subcategory* **PosComp** $\hookrightarrow$ **OrdCH** *is reflective.*
- *The unit interval* $[0,1]$ *is is an initial cogenerator on* **PosComp**.
- *The unit interval* $[0,1]$ *is injective in* **PosComp** *with respect to embeddings.*
- *The regular monomorphisms in* **PosComp** *are the embeddings, and the epimorphisms are the surjections.*
- $\mathbf{PosComp}^{\mathrm{op}}$ *is a quasivariety.*
- *The* $\aleph_1$*-copresentable objects in* **PosComp** *are precisely the metrisable partially ordered compact spaces. In particular,* $[0,1]$ *is* $\aleph_1$*-copresentable.*
- $\mathbf{PosComp}^{\mathrm{op}}$ *is a* $\aleph_1$*-quasivariety.*

Metric spaces as categories (again)

Examples

1. For $\overleftarrow{[0,\infty]}_+$ with $\otimes = +$ and $k = 0$:

$$\overleftarrow{[0,\infty]}_+\text{-}\mathbf{Cat} \simeq \mathbf{Met}.$$

$$\boxed{\begin{array}{l}\text{Metric space: } X \text{ with } a\colon X \times X \to [0,\infty] \text{ with} \\ \quad 0 \geq a(x,x) \qquad \text{and} \qquad d(x,y) + a(y,z) \geq d(x,z).\end{array}}$$

Metric spaces as categories (again)

Examples

1. For $\overleftarrow{[0,\infty]}_+$ with $\otimes = +$ and $k = 0$:
$$\overleftarrow{[0,\infty]}_+\text{-}\mathbf{Cat} \simeq \mathbf{Met}.$$
2. For $\overleftarrow{[0,\infty]}_\wedge$ with $\otimes = \max$ and $k = 0$:
$$\overleftarrow{[0,\infty]}_\wedge\text{-}\mathbf{Cat} \simeq \mathbf{UMet}.$$

Ultrametric space: X with $a\colon X \times X \to [0,\infty]$ with

$0 \geq a(x,x)$ and $max(d(x,y), a(y,z)) \geq d(x,z)$.

Metric spaces as categories (again)

Examples

1. For $\overleftarrow{[0,\infty]}_+$ with $\otimes = +$ and $k = 0$:
$$\overleftarrow{[0,\infty]}_+\text{-}\mathbf{Cat} \simeq \mathbf{Met}.$$
2. For $\overleftarrow{[0,\infty]}_\wedge$ with $\otimes = \max$ and $k = 0$:
$$\overleftarrow{[0,\infty]}_\wedge\text{-}\mathbf{Cat} \simeq \mathbf{UMet}.$$
3. For $\overleftarrow{[0,1]}_\oplus$ with $\otimes = \oplus$ and $k = 0$:
$$\overleftarrow{[0,1]}_\oplus\text{-}\mathbf{Cat} \simeq \mathbf{BMet}.$$

Metric spaces as categories (again)

Examples

1. For $\overleftarrow{[0,\infty]}_+$ with $\otimes = +$ and $k = 0$:
$$\overleftarrow{[0,\infty]}_+\text{-}\mathbf{Cat} \simeq \mathbf{Met}.$$
2. For $\overleftarrow{[0,\infty]}_\wedge$ with $\otimes = \max$ and $k = 0$:
$$\overleftarrow{[0,\infty]}_\wedge\text{-}\mathbf{Cat} \simeq \mathbf{UMet}.$$
3. For $\overleftarrow{[0,1]}_\oplus$ with $\otimes = \oplus$ and $k = 0$:
$$\overleftarrow{[0,1]}_\oplus\text{-}\mathbf{Cat} \simeq \mathbf{BMet}.$$
4. For $[0,1]_*$ with $\otimes = *$ and $k = 1$: $[0,1]_* \simeq \overleftarrow{[0,\infty]}_+$.

Metric spaces as categories (again)

Examples

1. For $\overleftarrow{[0,\infty]}_+$ with $\otimes = +$ and $k = 0$:
$$\overleftarrow{[0,\infty]}_+\text{-}\mathbf{Cat} \simeq \mathbf{Met}.$$
2. For $\overleftarrow{[0,\infty]}_\wedge$ with $\otimes = \max$ and $k = 0$:
$$\overleftarrow{[0,\infty]}_\wedge\text{-}\mathbf{Cat} \simeq \mathbf{UMet}.$$
3. For $\overleftarrow{[0,1]}_\oplus$ with $\otimes = \oplus$ and $k = 0$:
$$\overleftarrow{[0,1]}_\oplus\text{-}\mathbf{Cat} \simeq \mathbf{BMet}.$$
4. For $[0,1]_*$ with $\otimes = *$ and $k = 1$: $[0,1]_* \simeq \overleftarrow{[0,\infty]}_+$.
5. For $[0,1]_\wedge$ with $\otimes = \wedge$ and $k = 1$: $[0,1]_\wedge \simeq \overleftarrow{[0,\infty]}_\wedge$.

Metric spaces as categories (again)

Examples

1. For $\overleftarrow{[0,\infty]}_+$ with $\otimes = +$ and $k = 0$:
$$\overleftarrow{[0,\infty]}_+\text{-}\mathbf{Cat} \simeq \mathbf{Met}.$$
2. For $\overleftarrow{[0,\infty]}_\wedge$ with $\otimes = \max$ and $k = 0$:
$$\overleftarrow{[0,\infty]}_\wedge\text{-}\mathbf{Cat} \simeq \mathbf{UMet}.$$
3. For $\overleftarrow{[0,1]}_\oplus$ with $\otimes = \oplus$ and $k = 0$:
$$\overleftarrow{[0,1]}_\oplus\text{-}\mathbf{Cat} \simeq \mathbf{BMet}.$$
4. For $[0,1]_*$ with $\otimes = *$ and $k = 1$: $[0,1]_* \simeq \overleftarrow{[0,\infty]}_+$.
5. For $[0,1]_\wedge$ with $\otimes = \wedge$ and $k = 1$: $[0,1]_\wedge \simeq \overleftarrow{[0,\infty]}_\wedge$.
6. For $[0,1]_\odot$ with $u \otimes v = u + v - 1$ and $k = 1$:
$$[0,1]_\odot \simeq \overleftarrow{[0,1]}_\oplus.$$

Continuous quantale structures on $[0, 1]$

If `Time` $\gg$ 9h26m then skip

Continuous quantale structures on $[0, 1]$

Faucett (1955) and Mostert and Shields (1957)

consider continuous quantale structures $\otimes$ on $[0, 1]$ with neutral element 1.

Continuous quantale structures on $[0, 1]$

Faucett (1955) and Mostert and Shields (1957)

consider continuous quantale structures $\otimes$ on $[0, 1]$ with neutral element 1.

Proposition

Assume that 0 *and* 1 *are the only idempotent elements of* $[0, 1]$. *If*

Continuous quantale structures on $[0, 1]$

Faucett (1955) and Mostert and Shields (1957)

consider continuous quantale structures $\otimes$ on $[0, 1]$ with neutral element 1.

Proposition

Assume that 0 *and* 1 *are the only idempotent elements of* $[0, 1]$*. If*

1. $[0, 1]$ *has no nilpotent elements, then* $\otimes = *$ *is multiplication.*

Continuous quantale structures on $[0,1]$

Faucett (1955) and Mostert and Shields (1957)

consider continuous quantale structures $\otimes$ on $[0,1]$ with neutral element 1.

Proposition

Assume that 0 *and* 1 *are the only idempotent elements of* $[0,1]$*. If*

1. $[0,1]$ *has no nilpotent elements, then* $\otimes = *$ *is multiplication.*
2. $[0,1]$ *has a nilpotent element, then* $\otimes = \odot$ *is the Łukasiewicz tensor (and every element* x *with* $0 < x < 1$ *is nilpotent).*

Continuous quantale structures on $[0,1]$

Faucett (1955) and Mostert and Shields (1957)

consider continuous quantale structures $\otimes$ on $[0,1]$ with neutral element 1.

Proposition

Assume that 0 *and* 1 *are the only idempotent elements of* $[0,1]$. *If*

1. $[0,1]$ *has no nilpotent elements, then* $\otimes = *$ *is multiplication.*
2. $[0,1]$ *has a nilpotent element, then* $\otimes = \odot$ *is the Łukasiewicz tensor (and every element* x *with* $0 < x < 1$ *is nilpotent).*
3. *every element is idempotent, then* $\otimes = \wedge$.

Continuous quantale structures on $[0, 1]$

Faucett (1955) and Mostert and Shields (1957)

consider continuous quantale structures $\otimes$ on $[0, 1]$ with neutral element 1.

Proposition

Assume that 0 and 1 are the only idempotent elements of $[0, 1]$. If

1. *$[0, 1]$ has no nilpotent elements, then $\otimes = *$ is multiplication.*
2. *$[0, 1]$ has a nilpotent element, then $\otimes = \odot$ is the Łukasiewicz tensor (and every element x with $0 < x < 1$ is nilpotent).*
3. *every element is idempotent, then $\otimes = \wedge$.*

Theorem

For every non-idempotent $x \in [0, 1]$, there exist idempotent elements $e, f \in [0, 1]$, with $e < x < f$, such that the quantale $[e, f]$ is isomorphic to the quantale $[0, 1]$ with either multiplication or Łukasiewicz tensor.

Metric spaces vs. ordered sets

Forgetting something

The functor $[0,1]$-**Cat** $\longrightarrow$ **Ord** is defined by

$$x \leq y \text{ whenever } 1 \leq a(x,y).$$

Metric spaces vs. ordered sets

Forgetting something

The functor $[0,1]$-**Cat** $\longrightarrow$ **Ord** is defined by

$$x \leq y \text{ whenever } 1 \leq a(x,y).$$

Definition

A $[0,1]$-category (X,a) is called copowered whenever

$$a(x,-)\colon X \to [0,1] \text{ has a left adjoint } x \otimes -\colon [0,1] \to X$$

in $[0,1]$-**Cat**, for every $x \in X$.

Metric spaces vs. ordered sets

Forgetting something

The functor $[0,1]$-**Cat** $\longrightarrow$ **Ord** is defined by

$$x \le y \text{ whenever } 1 \le a(x,y).$$

Definition

A $[0,1]$-category (X,a) is called copowered whenever

$$a(x,-)\colon X \to [0,1] \text{ has a left adjoint } x \otimes -\colon [0,1] \to X$$

in $[0,1]$-**Cat**, for every $x \in X$.

This action of $[0,1]$ satisfies . . .

1. $x \otimes 1 = x$,
2. $(x \otimes u) \otimes v = x \otimes (u \otimes v)$,
3. $x \otimes \bigvee_{i \in I} u_i = \bigvee_{i \in I} (x \otimes u_i)$;

Metric spaces vs. ordered sets

This action of $[0, 1]$ satisfies ...

1. $x \otimes 1 = x$,
2. $(x \otimes u) \otimes v = x \otimes (u \otimes v)$,
3. $x \otimes \bigvee_{i \in I} u_i = \bigvee_{i \in I} (x \otimes u_i)$;

Metric spaces vs. ordered sets

This action of $[0,1]$ satisfies ...

1. $x \otimes 1 = x$,
2. $(x \otimes u) \otimes v = x \otimes (u \otimes v)$,
3. $x \otimes \bigvee_{i \in I} u_i = \bigvee_{i \in I} (x \otimes u_i)$;

Vice versa. . .

. . . every ordered set with such an action becomes a $[0,1]$-category:

$$\text{define } a\colon X \times X \to [0,1] \text{ by } x \otimes - \dashv a(x,-).$$

Metric spaces vs. ordered sets

This action of $[0,1]$ satisfies ...

1. $x \otimes 1 = x$,
2. $(x \otimes u) \otimes v = x \otimes (u \otimes v)$,
3. $x \otimes \bigvee_{i \in I} u_i = \bigvee_{i \in I} (x \otimes u_i)$;

Vice versa...

... every ordered set with such an action becomes a $[0,1]$-category:

$$\text{define } a \colon X \times X \to [0,1] \text{ by } x \otimes - \dashv a(x,-).$$

Finitely cocomplete metric space

$=$ ordered set with an action of $[0,1]$ with finite suprema preserved by the action.

Our setting (see Pedro's PhD thesis)

We consider:

$$\begin{tikzcd}
\mathbf{PosComp}_{\mathbb{V}} \arrow[rr, "C"] & & \mathrm{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}, \\
& \mathbf{PosComp} \arrow[ul] \arrow[ur, "C=\hom(-,{[0,1]}^{\mathrm{op}})"'] &
\end{tikzcd}$$

where, for $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ in $\mathbf{PosComp}_{\mathbb{V}}$,

$$C\varphi\colon CY \longrightarrow CX,\ \psi \longmapsto \left(x \mapsto \sup_{x\,\varphi\,y} \psi(y)\right).$$

Lax means

$$\Phi(1) \leq 1 \quad \text{and} \quad \Phi(\psi_1 \otimes \psi_2) \leq \Phi(\psi_1) \otimes \Phi(\psi_2).$$

Our setting (see Pedro's PhD thesis)

We consider:

$$\begin{tikzcd} \mathbf{PosComp}_{\mathbb{V}} \arrow[rr, "C"] & & \mathrm{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}, \\ & \mathbf{PosComp} \arrow[ul] \arrow[ur, "C=\mathrm{hom}(-,[0,1]^{\mathrm{op}})"'] & \end{tikzcd}$$

where, for $\varphi : X \nrightarrow Y$ in $\mathbf{PosComp}_{\mathbb{V}}$,

$$C\varphi \colon CY \longrightarrow CX,\ \psi \longmapsto \left(x \mapsto \sup_{x\,\varphi\,y} \psi(y)\right).$$

The induced monad morphism j is given by the family of maps

$$j_X \colon VX \longrightarrow [CX,[0,1]],\ A \longmapsto \Phi_A,$$

with $\Phi_A \colon CX \to [0,1],\ \psi \mapsto \sup_{x\in A} \psi(x)$.

Restricting to functions

Proposition

Let X be in **PosComp** $\sim$ **StablyComp** *and $A \subseteq X$ closed and upper. Then A is irreducible if and only if Φ_A satisfies*

$$\Phi_A(1) = 1 \quad \textit{and} \quad \Phi_A(\psi_1 \otimes \psi_2) = \Phi_A(\psi_1) \otimes \Phi_A(\psi_2).$$

Restricting to functions

Proposition

Let X be in **PosComp** $\sim$ **StablyComp** *and $A \subseteq X$ closed and upper. Then A is irreducible if and only if Φ_A satisfies*

$$\Phi_A(1) = 1 \quad \textit{and} \quad \Phi_A(\psi_1 \otimes \psi_2) = \Phi_A(\psi_1) \otimes \Phi_A(\psi_2).$$

Remark

Every stably compact space is sober.

Restricting to functions

Proposition

Let X be in **PosComp** $\sim$ **StablyComp** *and $A \subseteq X$ closed and upper. Then A is irreducible if and only if Φ_A satisfies*

$$\Phi_A(1) = 1 \quad \textit{and} \quad \Phi_A(\psi_1 \otimes \psi_2) = \Phi_A(\psi_1) \otimes \Phi_A(\psi_2).$$

Remark

Every stably compact space is sober.

Corollary

Let $\varphi \colon X \nrightarrow Y$ in $\mathbf{PosComp}_{\mathbb{V}}$. *Then φ is a function if and only if $C\varphi$ preserves 1 and $\otimes$.*

Some full embeddings

Theorem

For $\otimes = *$ or $\otimes = \odot$, the monad morphism j is an isomorphism. Therefore the functors

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \mathsf{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$$
$$C\colon \mathbf{PosComp} \longrightarrow \mathsf{Mon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$$

are fully faithful.

Some full embeddings

Theorem

For $\otimes = *$ or $\otimes = \odot$, the monad morphism j is an isomorphism. Therefore the functors

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \mathsf{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$$
$$C\colon \mathbf{PosComp} \longrightarrow \mathsf{Mon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$$

are fully faithful.

Can we do better?

Probably but

Some full embeddings

Theorem

For $\otimes = *$ or $\otimes = \odot$, the monad morphism j is an isomorphism. Therefore the functors

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \mathrm{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$$
$$C\colon \mathbf{PosComp} \longrightarrow \mathrm{Mon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$$

are fully faithful.

Can we do better?

Probably but

- For $\otimes = *, \odot$: $C\colon \mathbf{PosComp}_{\mathbb{V}} \to [0,1]\text{-}\mathbf{FinSup}^{\mathrm{op}}$ is not full.

  It is full for the "enriched" Vietoris monad.

Some full embeddings

Theorem

For $\otimes = *$ or $\otimes = \odot$, the monad morphism j is an isomorphism. Therefore the functors

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \mathsf{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$$
$$C\colon \mathbf{PosComp} \longrightarrow \mathsf{Mon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$$

are fully faithful.

Can we do better?

Probably but

- For $\otimes = *, \odot$: $C\colon \mathbf{PosComp}_{\mathbb{V}} \to [0,1]\text{-}\mathbf{FinSup}^{\mathrm{op}}$ is not full.
- $C\colon \mathbf{PosComp}_{\mathbb{V}} \to \mathsf{LaxMon}([0,1]_{\wedge}\text{-}\mathbf{FinSup})^{\mathrm{op}}$ is not full.

  $V1$ contains two elements; however, for every $\alpha \in [0,1]$, the map $\Phi = \alpha \wedge -\colon [0,1] \to [0,1]$ is in $\mathsf{LaxMon}([0,1]_{\wedge}\text{-}\mathbf{FinSup})$

Some full embeddings

Theorem

For $\otimes = *$ or $\otimes = \odot$, the monad morphism j is an isomorphism. Therefore the functors

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \mathsf{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$$
$$C\colon \mathbf{PosComp} \longrightarrow \mathsf{Mon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$$

are fully faithful.

Can we do better?

Probably but

- For $\otimes = *, \odot$: $C\colon \mathbf{PosComp}_{\mathbb{V}} \to [0,1]\text{-}\mathbf{FinSup}^{\mathrm{op}}$ is not full.
- $C\colon \mathbf{PosComp}_{\mathbb{V}} \to \mathsf{LaxMon}([0,1]_{\wedge}\text{-}\mathbf{FinSup})^{\mathrm{op}}$ is not full.
- $C\colon \mathbf{CompHaus}_{\mathbb{V}} \to \mathsf{LaxMon}([0,1]_{\wedge}\text{-}\mathbf{FinSup})^{\mathrm{op}}$ is not full.

Restricting to functions

Example

$C\colon \mathbf{PosComp} \longrightarrow \mathsf{Mon}([0,1]_\wedge\text{-}\mathbf{FinSup})^{\mathrm{op}}$ is not full.

Restricting to functions

Example

$C\colon \mathbf{PosComp} \longrightarrow \mathrm{Mon}([0,1]_\wedge\text{-}\mathbf{FinSup})^{\mathrm{op}}$ is not full.

For the separated ordered compact space $X = \{0 \leq 1\}$,

$$CX = \{(u,v) \in [0,1] \times [0,1] \mid u \leq v\}.$$

VX contains three elements; however, for every $\alpha \in [0,1]$, the map

$$\Phi_\alpha : CX \longrightarrow [0,1],\ (u,v) \longmapsto u \vee (\alpha \wedge v)$$

is in $\mathrm{Mon}([0,1]_\wedge\text{-}\mathbf{FinSup})$.

Restricting to functions

Example

$C\colon \mathbf{PosComp} \longrightarrow \mathrm{Mon}([0,1]_\wedge\text{-}\mathbf{FinSup})^{\mathrm{op}}$ is not full.

Theorem

$C\colon \mathbf{CompHaus} \longrightarrow \mathrm{Mon}([0,1]_\wedge\text{-}\mathbf{FinSup})^{\mathrm{op}}$ *is fully faithful.*

Bernhard Banaschewski. "On lattices of continuous functions". In: *Quaestiones Mathematicæ* **6**.(1-3) (1983), pp. 1–12.

Remark

Banaschewski does not consider $\mathrm{Mon}([0,1]_\wedge\text{-}\mathbf{FinSup})$ but the category of distributive lattices with constants from $[0,1]$.

Restricting to functions

Example

$C\colon \mathbf{PosComp} \longrightarrow \mathrm{Mon}([0,1]_\wedge\text{-}\mathbf{FinSup})^{\mathrm{op}}$ is not full.

Theorem

$C\colon \mathbf{CompHaus} \longrightarrow \mathrm{Mon}([0,1]_\wedge\text{-}\mathbf{FinSup})^{\mathrm{op}}$ *is fully faithful.*

Bernhard Banaschewski. "On lattices of continuous functions". In: *Quaestiones Mathematicæ* **6**.(1-3) (1983), pp. 1–12.

Theorem

$C\colon \mathbf{CompHaus} \longrightarrow \mathrm{Mon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$ *is fully faithful.*

Restricting to functions

Example

$C\colon \mathbf{PosComp} \longrightarrow \mathrm{Mon}([0,1]_{\wedge}\text{-}\mathbf{FinSup})^{\mathrm{op}}$ is not full.

Theorem

$C\colon \mathbf{CompHaus} \longrightarrow \mathrm{Mon}([0,1]_{\wedge}\text{-}\mathbf{FinSup})^{\mathrm{op}}$ *is fully faithful.*

Bernhard Banaschewski. "On lattices of continuous functions". In: *Quaestiones Mathematicæ* **6**.(1-3) (1983), pp. 1–12.

Theorem

$C\colon \mathbf{CompHaus} \longrightarrow \mathrm{Mon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$ *is fully faithful.*

Theorem

We consider an additional operation $\ominus$ in our theory (which is truncated minus in $[0,1]$).

Restricting to functions

Example

$C\colon \mathbf{PosComp} \longrightarrow \mathrm{Mon}([0,1]_\wedge\text{-}\mathbf{FinSup})^{\mathrm{op}}$ is not full.

Theorem

$C\colon \mathbf{CompHaus} \longrightarrow \mathrm{Mon}([0,1]_\wedge\text{-}\mathbf{FinSup})^{\mathrm{op}}$ *is fully faithful.*

Bernhard Banaschewski. "On lattices of continuous functions". In: *Quaestiones Mathematicæ* **6**.(1-3) (1983), pp. 1–12.

Theorem

$C\colon \mathbf{CompHaus} \longrightarrow \mathrm{Mon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$ *is fully faithful.*

Theorem

We consider an additional operation $\ominus$ *in our theory (which is truncated minus in* $[0,1]$*). Then*

$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \mathrm{LaxMon}_{\ominus}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$ *is fully faithful.*

Dual equivalences

Restricting the codomain of C

We consider only $\otimes = *$ or $\otimes = \odot$. At the codomain of

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \dots$$

we add

Dual equivalences

Restricting the codomain of C

We consider only $\otimes = *$ or $\otimes = \odot$. At the codomain of

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \dots$$

we add

- powers from $[0,1]$,

$$\boxed{a(-,x)\colon X^{\mathrm{op}} \to [0,1] \text{ has a left adjoint in } [0,1]\text{-}\mathbf{Cat}.}$$

Dual equivalences

Restricting the codomain of C

We consider only $\otimes = *$ or $\otimes = \odot$. At the codomain of

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \dots$$

we add

- powers from $[0,1]$,
- Cauchy completeness (*à la* Lawvere),

Dual equivalences

Restricting the codomain of C

We consider only $\otimes = *$ or $\otimes = \odot$. At the codomain of

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \dots$$

we add

- powers from $[0,1]$,
- Cauchy completeness (*à la* Lawvere),
- if $\otimes = *$: truncated minus $\ominus$ (unfortunately);

Restricting the codomain of C

We consider only $\otimes = *$ or $\otimes = \odot$. At the codomain of

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \dots$$

we add

- powers from $[0,1]$,
- Cauchy completeness (*à la* Lawvere),
- if $\otimes = *$: truncated minus $\ominus$ (unfortunately);

and morphisms preserving these additional operations.

Dual equivalences

Restricting the codomain of C

We consider only $\otimes = *$ or $\otimes = \odot$. At the codomain of

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \dots$$

we add

- powers from $[0,1]$,
- Cauchy completeness (*à la* Lawvere),
- if $\otimes = *$: truncated minus $\ominus$ (unfortunately);

and morphisms preserving these additional operations.

Question: Is the cone $(\varphi\colon A \to [0,1])_\varphi$ point-separating?

Recall: A lattice L is distributive iff the cone $(\varphi\colon L \to 2)_\varphi$ is point-separating.

Dual equivalences

Restricting the codomain of C

We consider only $\otimes = *$ or $\otimes = \odot$. At the codomain of

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \dots$$

we add

- powers from $[0,1]$,
- Cauchy completeness (*à la* Lawvere),
- if $\otimes = *$: truncated minus $\ominus$ (unfortunately);

and morphisms preserving these additional operations.

Question: Is the cone $(\varphi\colon A \to [0,1])_\varphi$ point-separating?

Answer: We don't know. If you do, please send it to `dirk@ua.pt`.

Dual equivalences

Restricting the codomain of C

We consider only $\otimes = *$ or $\otimes = \odot$. At the codomain of

$$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \dots$$

we add

- powers from $[0,1]$,
- Cauchy completeness (*à la* Lawvere),
- if $\otimes = *$: truncated minus $\ominus$ (unfortunately);

and morphisms preserving these additional operations.

Question: Is the cone $(\varphi\colon A \to [0,1])_\varphi$ point-separating?

Answer: We don't know. If you do, please send it to `dirk@ua.pt`.

Theorem

Restricting to those objects, $C\colon \mathbf{PosComp}_{\mathbb{V}} \to \dots$ becomes an equivalence.

A bit more general

Assumptions

We consider only $\otimes = *$ or $\otimes = \odot$. Moreover

A bit more general

Assumptions

We consider only $\otimes = *$ or $\otimes = \odot$. Moreover

- **PosComp** $\quad\rightsquigarrow\quad$ $\mathbf{MetCH}_{\mathrm{sep}}$.

A bit more general

Assumptions

We consider only $\otimes = *$ or $\otimes = \odot$. Moreover

- **PosComp** $\rightsquigarrow$ $\mathbf{MetCH}_{\mathrm{sep}}$.
- classical Vietoris $\rightsquigarrow$ enriched Vietoris.

  Recall: The elementos of VX are "approach maps" $\varphi\colon X \to [0,1]$ instead of closed subsets $A \subseteq X$ (that is, continuous maps $X \to 2$).

A bit more general

Assumptions

We consider only $\otimes = *$ or $\otimes = \odot$. Moreover

- **PosComp** $\rightsquigarrow$ $\mathbf{MetCH}_{\mathrm{sep}}$.
- classical Vietoris $\rightsquigarrow$ enriched Vietoris.

The setting

$$\begin{tikzcd}
(\mathbf{MetCH}_{\mathrm{sep}})_{\mathbb{V}} \arrow[rr, "C=\mathrm{hom}(-{,}1)"] & & {[0,1]\text{-}\mathbf{FinSup}^{\mathrm{op}}} \\
& \mathbf{MetCH}_{\mathrm{sep}} \arrow[ul] \arrow[ur, swap, "C=\mathrm{hom}(-{,}[0{,}1]^{\mathrm{op}})"] &
\end{tikzcd}$$

induces the monad morphism

$$j_X \colon VX \longrightarrow [CX, [0,1]], \; (\varphi \colon 1 \mathrel{\circ\!\!\rightarrow} X) \longmapsto (\psi \mapsto \psi \cdot \varphi).$$

Metric compact Hausdorff spaces

Question

Is $[0,1]^{\mathrm{op}}$ an initial cogenerator in $\mathbf{MetCH}_{\mathrm{sep}}$?

We don't know. Please send the answer...

Metric compact Hausdorff spaces

Question

Is $[0,1]^{\mathrm{op}}$ an initial cogenerator in $\mathbf{MetCH}_{\mathrm{sep}}$?

We don't know. Please send the answer...

Remark

$[0,1]^{\mathrm{op}} \not\simeq [0,1]$ in $\mathbf{MetCH}$.

Question

Is $[0,1]^{\mathrm{op}}$ an initial cogenerator in $\mathbf{MetCH}_{\mathrm{sep}}$?

We don't know. Please send the answer...

Proposition

X is $[0,1]^{\mathrm{op}}$-cogenerated $\implies$ VX is $[0,1]^{\mathrm{op}}$-cogenerated.

Metric compact Hausdorff spaces

Question

Is $[0,1]^{\mathrm{op}}$ an initial cogenerator in $\mathbf{MetCH}_{\mathrm{sep}}$?

We don't know. Please send the answer...

Proposition

X is $[0,1]^{\mathrm{op}}$-cogenerated $\implies$ VX is $[0,1]^{\mathrm{op}}$-cogenerated.

Notation

$\mathbf{MetCH}_{[0,1]^{\mathrm{op}}}$ = the full subcategory of $\mathbf{MetCH}$ defined by $[0,1]^{\mathrm{op}}$-cogenerated objects.

Metric compact Hausdorff spaces

Question

Is $[0,1]^{\mathrm{op}}$ an initial cogenerator in $\mathbf{MetCH}_{\mathrm{sep}}$?

We don't know. Please send the answer. . .

Proposition

X is $[0,1]^{\mathrm{op}}$-cogenerated $\implies$ VX is $[0,1]^{\mathrm{op}}$-cogenerated.

Notation

$\mathbf{MetCH}_{[0,1]^{\mathrm{op}}}$ = the full subcategory of $\mathbf{MetCH}$ defined by $[0,1]^{\mathrm{op}}$-cogenerated objects.

Every partially ordered compact space is $[0,1]^{\mathrm{op}}$-cogenerated.

Metric compact Hausdorff spaces

Question

Is $[0,1]^{\mathrm{op}}$ an initial cogenerator in $\mathbf{MetCH}_{\mathrm{sep}}$?

We don't know. Please send the answer...

Proposition

X is $[0,1]^{\mathrm{op}}$-cogenerated $\implies$ VX is $[0,1]^{\mathrm{op}}$-cogenerated.

Notation

$\mathbf{MetCH}_{[0,1]^{\mathrm{op}}}$ = the full subcategory of $\mathbf{MetCH}$ defined by $[0,1]^{\mathrm{op}}$-cogenerated objects.

Theorem

The functor

$$C\colon \left(\mathbf{MetCH}_{[0,1]^{\mathrm{op}}}\right)_{\mathbb{V}} \longrightarrow [0,1]\text{-}\mathbf{FinSup}^{\mathrm{op}}$$

is fully faithful.

Restricting to functions

Before

$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \mathrm{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$

- $A \subseteq X$ closed $\quad \leftrightsquigarrow \quad \Phi : CX \longrightarrow [0,1].$

Now

$C\colon (\mathbf{MetCH}_{[0,1]^{\mathrm{op}}})_{\mathbb{V}} \longrightarrow ([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$

- $\varphi\colon X \longrightarrow [0,1] \quad \leftrightsquigarrow \quad \Phi : CX \longrightarrow [0,1].$

Restricting to functions

Before

$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \mathsf{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$

- $A \subseteq X$ closed $\quad\leftrightsquigarrow\quad$ $\Phi : CX \longrightarrow [0,1]$.
- A is irreducible $\quad\iff\quad$ Φ is in $\mathsf{Mon}([0,1]\text{-}\mathbf{FinSup})$.

Now

$C\colon (\mathbf{MetCH}_{[0,1]^{\mathrm{op}}})_{\mathbb{V}} \longrightarrow ([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$

- $\varphi\colon X \longrightarrow [0,1]$ $\quad\leftrightsquigarrow\quad$ $\Phi : CX \longrightarrow [0,1]$.

Restricting to functions

Before

$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \mathrm{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$

- $A \subseteq X$ closed $\quad\leftrightsquigarrow\quad \Phi : CX \longrightarrow [0,1].$
- A is irreducible $\quad\Longleftrightarrow\quad \Phi$ is in $\mathrm{Mon}([0,1]\text{-}\mathbf{FinSup})$.

Now

$C\colon (\mathbf{MetCH}_{[0,1]^{\mathrm{op}}})_{\mathbb{V}} \longrightarrow ([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$

- $\varphi\colon X \longrightarrow [0,1] \quad\leftrightsquigarrow\quad \Phi : CX \longrightarrow [0,1].$
- $1 \xrightarrow{\varphi}\!\!\!\!\!\!\circ\;\; X$ is irreducible(?) $\quad\Longleftrightarrow\quad \Phi$ is ????

Restricting to functions

Before

$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \mathsf{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$

- $A \subseteq X$ closed $\quad\leftrightsquigarrow\quad$ $\Phi : CX \longrightarrow [0,1]$.
- A is irreducible $\quad\Longleftrightarrow\quad$ Φ is in $\mathsf{Mon}([0,1]\text{-}\mathbf{FinSup})$.
- Every X in **StablyComp** is a sober space.

Now

$C\colon (\mathbf{MetCH}_{[0,1]^{\mathrm{op}}})_{\mathbb{V}} \longrightarrow ([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$

- $\varphi\colon X \longrightarrow [0,1]$ $\quad\leftrightsquigarrow\quad$ $\Phi : CX \longrightarrow [0,1]$.
- $1 \overset{\varphi}{\multimap} X$ is irreducible(?) $\quad\Longleftrightarrow\quad$ Φ is ????

Restricting to functions

Before

$C\colon \mathbf{PosComp}_{\mathbb{V}} \longrightarrow \mathsf{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$

- $A \subseteq X$ closed $\quad\leftrightsquigarrow\quad$ $\Phi : CX \longrightarrow [0,1]$.
- A is irreducible $\quad\iff\quad$ Φ is in $\mathsf{Mon}([0,1]\text{-}\mathbf{FinSup})$.
- Every X in **StablyComp** is a sober space.

Now

$C\colon (\mathbf{MetCH}_{[0,1]^{\mathrm{op}}})_{\mathbb{V}} \longrightarrow ([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$

- $\varphi\colon X \longrightarrow [0,1]$ $\quad\leftrightsquigarrow\quad$ $\Phi : CX \longrightarrow [0,1]$.
- $1 \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\circ\ } X$ is irreducible(?) $\quad\iff\quad$ Φ is ????
- Every X in **MetCH** is a sober(?) approach space ???

Cauchy complete approach spaces

Distributors

For approach spaces X and Y, a distributor $\varphi\colon X \looparrowright Y$ is a map $\varphi\colon UX \times Y \to [0,1]$ so that ....

Cauchy complete approach spaces

Distributors

For approach spaces X and Y, a distributor $\varphi\colon X \multimap Y$ is a map $\varphi\colon UX \times Y \to [0,1]$ so that ....

- $\varphi\colon 1 \multimap Y \quad = \quad$ approach map $\varphi\colon Y \to [0,1]$.

Cauchy complete approach spaces

Distributors

For approach spaces X and Y, a distributor $\varphi\colon X \,{\circ\!\!\rightharpoondown}\, Y$ is a map $\varphi\colon UX \times Y \to [0,1]$ so that ....

- $\varphi\colon 1 \,{\circ\!\!\rightharpoondown}\, Y \quad = \quad$ approach map $\varphi\colon Y \to [0,1]$.
- $\psi\colon X \,{\circ\!\!\rightharpoondown}\, 1 \quad = \quad$ approach map $\psi\colon (UX)^{\mathrm{op}} \to [0,1]$.

Cauchy complete approach spaces

Distributors

For approach spaces X and Y, a distributor $\varphi\colon X \,{\circ\!\!\!\rightharpoonup}\, Y$ is a map $\varphi\colon UX \times Y \to [0,1]$ so that ....

- $\varphi\colon 1 \,{\circ\!\!\!\rightharpoonup}\, Y \quad = \quad$ approach map $\varphi\colon Y \to [0,1]$.
- $\psi\colon X \,{\circ\!\!\!\rightharpoonup}\, 1 \quad = \quad$ approach map $\psi\colon (UX)^{\mathrm{op}} \to [0,1]$.

Definition

X is Cauchy complete if every adjunction $\varphi \dashv \psi$ is induced by some $x \in X$.[a] (that is: $\varphi = d(\{x\}, -)$)

[a] Maria Manuel Clementino and Dirk Hofmann. "Lawvere completeness in topology". In: *Applied Categorical Structures* **17**.(2) (2009), pp. 175–210.

Cauchy complete approach spaces

Distributors

For approach spaces X and Y, a distributor $\varphi\colon X \looparrowright Y$ is a map $\varphi\colon UX \times Y \to [0,1]$ so that ....

- $\varphi\colon 1 \looparrowright Y \quad = \quad$ approach map $\varphi\colon Y \to [0,1]$.
- $\psi\colon X \looparrowright 1 \quad = \quad$ approach map $\psi\colon (UX)^{\mathrm{op}} \to [0,1]$.

Definition

X is Cauchy complete if every adjunction $\varphi \dashv \psi$ is induced by some $x \in X$. (that is: $\varphi = d(\{x\},-)$)

Examples

- In **Top**: Cauchy complete = sober.

Cauchy complete approach spaces

Distributors

For approach spaces X and Y, a distributor $\varphi\colon X \multimapdotinv Y$ is a map $\varphi\colon UX \times Y \to [0,1]$ so that . . . .

- $\varphi\colon 1 \multimapdotinv Y \quad = \quad$ approach map $\varphi\colon Y \to [0,1]$.
- $\psi\colon X \multimapdotinv 1 \quad = \quad$ approach map $\psi\colon (UX)^{\mathrm{op}} \to [0,1]$.

Definition

X is Cauchy complete if every adjunction $\varphi \dashv \psi$ is induced by some $x \in X$. (that is: $\varphi = d(\{x\},-)$)

Examples

- In **Top**: Cauchy complete = sober.
- In **App**: Cauchy complete = approach sober[a].

[a] Bernhard Banaschewski, Robert Lowen, and Cristophe Van Olmen. "Sober approach spaces". In: *Topology and its Applications* **153**.(16) (2006), pp. 3059–3070.

Cauchy complete approach spaces

Distributors

For approach spaces X and Y, a distributor $\varphi\colon X \,{\circ\!\!\rightharpoonup}\, Y$ is a map $\varphi\colon UX \times Y \to [0,1]$ so that ....

- $\varphi\colon 1 \,{\circ\!\!\rightharpoonup}\, Y \quad = \quad$ approach map $\varphi\colon Y \to [0,1]$.
- $\psi\colon X \,{\circ\!\!\rightharpoonup}\, 1 \quad = \quad$ approach map $\psi\colon (UX)^{\mathrm{op}} \to [0,1]$.

Definition

X is Cauchy complete if every adjunction $\varphi \dashv \psi$ is induced by some $x \in X$. (that is: $\varphi = d(\{x\}, -)$)

Examples

- In **Top**: Cauchy complete = sober.
- In **App**: Cauchy complete = approach sober .

Proposition

Every metric compact Hausdorff space is Cauchy complete.

Adjoint distributors

Proposition

The following are equivalent.[a]

(i) $\varphi\colon 1 \;{-\!\!\!\circ\!\!\!\rightharpoonup}\; X$ *is left adjoint.*

[a] Dirk Hofmann and Isar Stubbe. "Towards Stone duality for topological theories". In: *Topology and its Applications* **158**.(7) (2011), pp. 913–925.

Adjoint distributors

Proposition

The following are equivalent.[a]

(i) $\varphi\colon 1 \,{\circ\!\!\!\rightharpoonup}\, X$ *is left adjoint.*

(ii) The metric map $[\varphi, -]\colon \mathbf{App}(X, [0,1]) \to [0,1]$ *preserves tensors and suprema (continuously) indexed by compact Hausdorff spaces.*

[a] Dirk Hofmann and Isar Stubbe. "Towards Stone duality for topological theories". In: *Topology and its Applications* **158**.(7) (2011), pp. 913–925.

Adjoint distributors

Proposition

The following are equivalent.[a]

(i) $\varphi\colon 1 \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ *is left adjoint.*

(ii) *The metric map* $[\varphi, -]\colon \mathbf{App}(X, [0,1]) \to [0,1]$ *preserves tensors and suprema (continuously) indexed by compact Hausdorff spaces.*

Not what one expects!! For a topological space, $A \subseteq X$ *is irreducible iff*

$$[A \subseteq -]\colon \mathbf{Top}(X, 2) \to 2$$

preserves finite suprema.

[a] Dirk Hofmann and Isar Stubbe. "Towards Stone duality for topological theories". In: *Topology and its Applications* **158**.(7) (2011), pp. 913–925.

Adjoint distributors

Proposition

The following are equivalent.[a]

(i) $\varphi\colon 1 \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ *is left adjoint.*

(ii) *The metric map* $[\varphi, -]\colon \mathbf{App}(X, [0,1]) \to [0,1]$ *preserves tensors and suprema (continuously) indexed by compact Hausdorff spaces.*

(iii) *The metric map* $[\varphi, -]\colon \mathbf{App}(X, [0,1]) \to [0,1]$ *preserves tensors and finite suprema.*[b]

---

[a] Dirk Hofmann and Isar Stubbe. "Towards Stone duality for topological theories". In: *Topology and its Applications* **158**.(7) (2011), pp. 913–925.

[b] Leopoldo Nachbin. "Compact unions of closed subsets are closed and compact intersections of open subsets are open". In: *Portugaliæ Mathematica* **49**.(4) (1992), pp. 403–409.

Adjoint distributors

Proposition

The following are equivalent.[a]

(i) $\varphi\colon 1 \,{\circ\!\!\rightharpoondown}\, X$ *is left adjoint.*

(ii) *The metric map* $[\varphi, -]\colon \mathbf{App}(X, [0,1]) \to [0,1]$ *preserves tensors and suprema (continuously) indexed by compact Hausdorff spaces.*

(iii) *The metric map* $[\varphi, -]\colon \mathbf{App}(X, [0,1]) \to [0,1]$ *preserves tensors and finite suprema.*

[a] Dirk Hofmann and Isar Stubbe. "Towards Stone duality for topological theories". In: *Topology and its Applications* **158**.(7) (2011), pp. 913–925.

Remark

This is not what we need. We wish to study the map $\varphi \cdot -$ instead of $[\varphi, -]$.

Restriction further

Assumption

We consider only the Łukasiewicz tensor $\otimes = \odot$

Restriction further

Assumption

We consider only the Łukasiewicz tensor $\otimes = \odot$ ... because it is a Girard quantale: for every $u \in [0,1]$,

$$u = \hom(\hom(u, \bot), \bot) \text{ where } \hom(u, \bot) = 1 - u =: u^{\bot}.$$

Restriction further

Assumption

We consider only the Łukasiewicz tensor $\otimes = \odot$ ... because it is a Girard quantale: for every $u \in [0,1]$,

$$u = \hom(\hom(u, \bot), \bot) \text{ where } \hom(u, \bot) = 1 - u =: u^{\bot}.$$

Why is that useful?

$$\begin{tikzcd}
{[0,1]\text{-}\mathbf{Dist}(X,1)} \arrow[r, "(-)^{\perp}"] \arrow[d, "(-\cdot\varphi)"'] & {[0,1]\text{-}\mathbf{Dist}(1,X)^{\mathrm{op}}} \arrow[d, "{[\varphi,-]^{\mathrm{op}}}"] \\
{[0,1]} \arrow[r, "(-)^{\perp}"'] & {[0,1]^{\mathrm{op}}}
\end{tikzcd}$$

commutes in $[0,1]$-**Cat**

Restriction further

Assumption

We consider only the Łukasiewicz tensor $\otimes = \odot$ ...because it is a Girard quantale: for every $u \in [0,1]$,

$$u = \hom(\hom(u,\bot),\bot) \text{ where } \hom(u,\bot) = 1 - u =: u^\perp.$$

Why is that useful?

$$\begin{tikzcd}
\mathbf{App}(X,[0,1]^{\mathrm{op}}) \arrow[r, "(-)^\perp"] \arrow[d, "(-\cdot\varphi)"'] & \mathbf{App}(X,[0,1])^{\mathrm{op}} \arrow[d, "{[\varphi,-]^{\mathrm{op}}}"] \\
{[0,1]} \arrow[r, "(-)^\perp"'] & {[0,1]^{\mathrm{op}}}
\end{tikzcd}$$

commutes in $[0,1]$-**Cat**

Restriction further

Assumption

We consider only the Łukasiewicz tensor $\otimes = \odot$ ... because it is a Girard quantale: for every $u \in [0,1]$,

$$u = \hom(\hom(u,\bot),\bot) \text{ where } \hom(u,\bot) = 1 - u =: u^{\bot}.$$

Why is that useful?

$$\begin{tikzcd} CX \arrow[r, hook] \arrow[dr, "\Phi"'] & \mathbf{App}(X,[0,1]^{\mathrm{op}}) \arrow[r, "(-)^{\bot}"] \arrow[d, "(-\cdot\varphi)"'] & \mathbf{App}(X,[0,1])^{\mathrm{op}} \arrow[d, "{[\varphi,-]^{\mathrm{op}}}"] \\ & {[0,1]} \arrow[r, "(-)^{\bot}"'] & {[0,1]^{\mathrm{op}}} \end{tikzcd}$$

commutes in $[0,1]$-**Cat** and $CX \hookrightarrow \mathbf{App}(X,[0,1]^{\mathrm{op}})$ is $\bigvee$-dense.

Putting it together

Assumption

We still consider only the Łukasiewicz tensor $\otimes = \odot$,

Theorem

$\varphi\colon 1 \nrightarrow X$ is left adjoint $\iff$ Φ preserves finite weighted limits.

Putting it together

Assumption

We still consider only the Łukasiewicz tensor $\otimes = \odot$,

Theorem

$\varphi\colon 1 \mathrel{\circ\!\!\!\!\rightharpoonup} X$ is left adjoint $\iff$ Φ preserves finite weighted limits.

Corollary

The fully faithful functor

$$C\colon (\mathbf{MetCH}_{[0,1]^{\mathrm{op}}})_{\mathbb{V}} \longrightarrow [0,1]\text{-}\mathbf{FinSup}^{\mathrm{op}}$$

restricts to a fully faithful functor

$$C\colon \mathbf{MetCH}_{[0,1]^{\mathrm{op}}} \longrightarrow [0,1]\text{-}\mathbf{FinLat}^{\mathrm{op}}.$$