## Boundedness in frames

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# Boundedness in topology

Boundedness is of course not a topological notion. Classic topological approximations have been via (relative) compactness.

A subspace  $A \subseteq X$  of a topological space X has been termed:

- ► Absolutely bounded (Gagola and Gemignani 1968) if A is contained in a member of any directed open cover of X.
- e-relatively compact (Hechler 1975) if any open cover C of A contains a finite subcover of A.
- Bounded (Lambrinos 1973 & 1976) if any open cover C of X contains a finite subcover of A.

#### Remark

From a topologist's point of view, it would seem desirable to have that A is bounded if and only if  $\overline{A}$  is bounded.

# Some terminology in frames

A frame is a complete lattice L with top 1, bottom 0 and distributivity

$$a \land \bigvee S = \bigvee_{s \in S} (a \land s).$$

• The **pseudocomplement** of  $a \in L$  is  $a^*$  defined by

$$x \leq a^* \Leftrightarrow x \wedge a = 0.$$

Additional order relations defined on L:

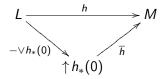
- ▶ **Rather below:**  $a \prec b$  iff there exists  $c \in L$  with  $a \land c = 0$  and  $b \lor c = 1$  iff  $a^* \lor b = 1$ .
- ▶ Completely below:  $a \prec d$  iff there exists  $\{c_r \mid r \in [0,1] \cap \mathbb{Q}\} \subseteq L$  with  $c_0 = a$ ,  $c_1 = b$  and  $c_r \prec c_s$  for any r < s.
- Way below: a ≪ b iff whenever b ≤ ∨ S then there exists finite A ⊆ S with a ≤ ∨ A.

A frame L is:

. . .

- **Regular** if  $a = \bigvee \{x \mid x \prec a\}$  for all  $a \in L$ .
- Completely regular if  $a = \bigvee \{x \mid x \prec a\}$  for all  $a \in L$ .
- Continuous if  $a = \bigvee \{x \mid x \ll a\}$  for all  $a \in L$ .

A frame homomorphism  $h: L \to M$  preserves  $\land$  and  $\bigvee$ . The right adjoint is denoted by  $h_*$ . Note that any  $h: L \to M$  factors through  $\uparrow h_*(0)$ ,



# Bounded elements in a frame

### Definition

An element  $a \in L$  is **bounded** if for any cover C of L,  $a^* \in C \Rightarrow C$  contains a finite subcover.

### Remarks

- Since  $a^* = a^{***}$ , *a* is bounded iff  $a^{**}$  is bounded.
- The set of all bounded elements Bd(L) forms an ideal in L.
- ▶ 1 is bounded iff *L* is compact.
- If a is bounded then  $a \ll 1$ .
- If L is regular then a is bounded iff  $a \ll 1$ .
- If  $\bigvee Bd(L) = 1$  then *a* is bounded iff  $a \ll 1$ .

# Bounded sublocales

## Definition (Dube 2005)

An onto map  $h: L \to M$  is a **bounded sublocale** of L if any cover C of L contains a finite K such that h[K] covers M.

## Proposition

- 1. An element  $a \in L$  is bounded iff  $\lor a^* : L \rightarrow \uparrow a^*$  is a bounded sublocale.
- 2. An element  $a \ll 1$  in L iff  $\land a : L \rightarrow \downarrow a$  is a bounded sublocale.

# Boundedness and filters

We say that a filter F on L clusters if  $\bigvee_{x \in F} x^* \neq 1$  and that F is **convergent** if F intersects every cover of L.

## Proposition

Consider the following properties of  $a \in L$ .

- 1. a is bounded.
- 2.  $a \ll 1$
- 3. For all filters F on L,  $a \in F \rightarrow F$  clusters.
- 4. For all filters F on L,  $a^* \notin F \Rightarrow F$  clusters.

5. For all prime filters F on L,  $a \in F \Rightarrow F$  is convergent. Then  $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4)$  and  $(2) \Rightarrow (5)$ . If L is regular then  $(4) \Rightarrow (1)$ .

# Bounded homomorphisms

## Definition

A homomorphism  $h: L \to M$  is **bounded** if there exists  $a \in Bd(L)$  with h(a) = 1.

### Remarks

- An obvious option is to consider h to be bounded if its image is a bounded sublocale. We call such h D-bounded, i.e. h for which any cover C of L contains a finite K such that h[K] covers M.
- In general if h is bounded then it is easily seen to be D-bounded. In the absence of additional assumptions on the frames or on Bd(L) it is not possible to extract a generic bounded element from a D-bounded map.

# Bounded homomorphisms...

# Proposition If $h: L \to M$ is bounded then $h_*(0)^*$ is bounded.

Lemma If  $h: L \to M$  with h(x) = 1 and  $x \prec y$  then  $h_*(0)^* \leq y$ .

## Proposition

In regular frames, if  $h: L \to M$  is D-bounded then  $h_*(0)^*$  is bounded.

## Corollary

In regular frames, if  $h: L \to M$  is a bounded (hence D-bounded) dense quotient then L is compact.

### Proposition

If  $\bigvee Bd(L) = 1$  then  $h: L \to M$  is bounded iff h is D-bounded.

# Pseudocompactness?

### Definition

Let *E* be a fixed frame. *L* is *E*-**pseudocompact** if every  $h: E \rightarrow M$  is bounded.

### Remarks

- If  $E = \mathcal{L}(\mathbb{R})$  this is the usual pseudo-compactness.
- ► The case E = P(N) was studied (briefly) by Marcus for completely regular frames.
- Understandably the study of pseudocompactness is restricted to frames with a degree of structure linked to the frame *E*. (Typically completely regular frames, *σ*-frames, *κ*-frames.)
- If  $\bigvee Bd(E) = 1$  then L compact  $\Rightarrow L$  is E-pseudocompact.

## References

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