Sobriety and congruence biframes

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Overview

- Sober spaces are the only topological spaces that can be faithfully represented by frames.
- But strictly zero-dimensional biframes can represent *all* T_0 spaces.
- So in that setting sobriety is a nontrivial property.

- A T_0 space X is sober iff these equivalent conditions hold:
    - Every irreducible closed set is the closure of a discrete subspace.
    - X is universally Skula-closed.
    - X is bicomplete in the well-monotone quasi-uniformity.[1]
- We will see that congruence biframes have analogous characterisations amongst strictly zero-dimensional biframes.

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Congruence frames

- The quotients of a frame L can be represented by their kernel equivalence relations, which are called congruences. This correspondence is *order-reversing*.
- That lattice $\mathbb{C}L$ of all congruences on L is itself a frame.
- A congruence ∇_a which induces a closed quotient is called a closed congruence. These form a subframe of $\mathbb{C}L$ isomorphic to L.
- Each closed congruence has a complement in $\mathbb{C}L$, which is called an open congruence.
- Together the closed and open congruences generate $\mathbb{C}L$.

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Strictly zero-dimensional biframes

- A biframe $\mathcal{L}$ is a triple $(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ where $\mathcal{L}_0$ is a frame and $\mathcal{L}_1$ and $\mathcal{L}_2$ are subframes of $\mathcal{L}_0$ which together generate $\mathcal{L}_0$.
- $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_0$ are called the first, second and total parts of $\mathcal{L}$.
- A biframe homomorphism $f\colon \mathcal{L} \to \mathcal{M}$ is a frame homomorphism $f_0\colon \mathcal{L}_0 \to \mathcal{M}_0$ which restricts to maps $f_i\colon \mathcal{L}_i \to \mathcal{M}_i$.
- The congruence frame has a biframe structure $(\mathbb{C}L, \nabla L, \Delta L)$, where ∇L is the subframe of closed congruences and ΔL is a subframe generated by the open congruences.
- The congruence biframe satisfies the following conditions.
  1) Every element of ∇L has a complement which lies in ΔL.
  2) ΔL is generated by these complements.

  We call such a biframe strictly zero-dimensional.

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Skula biframes

- We can get other examples of strictly zero-dimensional biframes from topological spaces.
- Let (X, τ) be a T_0 space. Let υ be the topology generated by taking the closed sets as open. The Skula topology σ is the join of τ and υ. We call (σ, τ, υ) the Skula biframe of (X, τ).
- Skula biframes are the *spatial* strictly zero-dimensional biframes.
- We obtain a fully faithful functor $\mathrm{Sk}\colon \mathrm{Top_0}^{\mathrm{op}} \to \mathrm{Str0DBiFrm}$, which is right adjoint to the functor $\Sigma_1\colon \mathrm{Str0DBiFrm} \to \mathrm{Top_0}^{\mathrm{op}}$ that sends $\mathcal{L}$ to the set of points of $\mathcal{L}_0$ equipped with the topology of $\mathcal{L}_1$.

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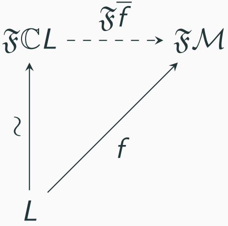
The universal property of congruence biframes

- There is an obvious forgetful functor $\mathfrak{F}\colon \mathrm{Str0DBiFrm} \to \mathrm{Frm}$ which takes first parts.
- The congruence biframe gives a functor that is left adjoint to $\mathfrak{F}$.



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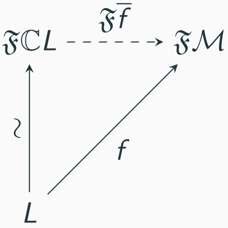

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The congruential coreflection as an analogue of sobrification

- $\Sigma_1(\mathbb{C}\mathfrak{F})\mathrm{Sk}$ is the sobrification functor and so sobrification appears as the 'spatial shadow' of the congruential coreflection.

$$\begin{array}{ccc}
\mathrm{Str0DBiFrm} & \xrightarrow{\mathbb{C}\mathfrak{F}} & \mathrm{Str0DBiFrm} \\
\mathrm{Sk}\uparrow & & \downarrow\Sigma_1 \\
\mathrm{Top_0}^{\mathrm{op}} & \xrightarrow[\mathrm{sob}^{\mathrm{op}}]{} & \mathrm{Top_0}^{\mathrm{op}}
\end{array}$$

- Here the functors Sk and Σ_1 are used to transport spaces into the setting of strictly zero-dimensional biframes and back.
- Note that $\Sigma_1\mathrm{Sk}$ is naturally isomorphic to the identity functor.

Dense quotients and universal closedness

- A biframe map f between strictly zero-dimensional biframes is surjective iff f_1 is surjective and dense iff f_1 is injective.
- So $\chi_{\mathcal{M}} \colon \mathbb{C}\mathcal{M}_1 \to \mathcal{M}$ is a dense surjection and every strictly zero-dimensional biframe is a dense quotient of a congruence biframe.

Lemma

Congruence biframes are precisely the universally closed strictly zero-dimensional biframes.

Proof.

If $\mathcal{M}$ is universally closed, then $\chi_{\mathcal{M}} \colon \mathbb{C}\mathcal{M}_1 \to \mathcal{M}$ is an isomorphism. Conversely, if $f \colon \mathcal{M} \twoheadrightarrow \mathbb{C}L$ is a dense surjection, then $\mathfrak{F}f$ is an iso. Hence, $f\chi_{\mathcal{M}} = \mathbb{C}\mathfrak{F}f$ is also an iso. But then f is a split bimorphism and therefore an isomorphism. □

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Permissible quotients

- Let $\mathcal{L}$ be a strictly zero-dimensional biframe. Since the right adjoint χ_* of the congruential coreflection $\chi_{\mathcal{L}}$ is injective, we can view elements of $\mathcal{L}$ as certain congruences on $\mathcal{L}_1$.

Proposition

For any $a \in \mathcal{L}_0$, we have $\mathfrak{F}(\mathcal{L}/\nabla_a) \cong \mathcal{L}_1/\chi_*(a)$.

- So the elements of $\mathcal{L}_0$ can be thought of as the 'permissible' quotients of $\mathcal{L}_1$.
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Clear elements of strictly zero-dimensional biframes

- Let $\mathcal{M}$ be a strictly zero-dimensional biframe. The closure of an element $a \in \mathcal{M}_0$ is the largest element $c\ell(a)$ of $\mathcal{M}_1$ lying below a.
- (Recall the order in $\mathbb{C}L$ is the reverse of the lattice of quotients.)
- Due to the existence of smallest dense sublocales, there is always a largest element of $\mathbb{C}L$ with a given closure.
- Such an element might not exist in a general strictly zero-dimensional biframe. When it does, we call this element clear and its closure clarifiable.

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A characterisation of congruence biframes via clear elements

Lemma

Let $\mathcal{L}$ be strictly zero-dimensional and $a \in \mathcal{L}_0$. Then a is clear iff $\chi_*(a)$ is a clear congruence iff the first part of $\mathcal{L}/\nabla_a$ is Boolean.

Corollary

In a Skula biframe $\mathrm{Sk}\,X$, an element $U \in (\mathrm{Sk}\,X)_1$ is clarifiable iff the closed subspace U^c is the closure of a discrete subspace. In particular, every prime element of $(\mathrm{Sk}\,X)_1$ is clarifiable iff X is sober.

Theorem

A strictly zero-dimensional biframe $\mathcal{L}$ is a congruence biframe iff all its closed elements are clarifiable.

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Quasi-uniform biframes

- A paircover U on a biframe $\mathcal{L}$ is a downset on $\mathcal{L}_1 \times \mathcal{L}_2$ that satisfies $\bigvee_{(x,y)\in U} x \wedge y = 1$.
- A quasi-uniform biframe $(\mathcal{L}, \mathcal{U})$ is a biframe $\mathcal{L}$ equipped with a filter $\mathcal{U}$ of paircovers satisfying certain axioms.
- A quasi-uniform biframe is bicomplete if whenever it is a quasi-uniform quotient of another quasi-uniform biframe, the quotient is a closed quotient.
- Every quasi-uniform biframe has a unique bicompletion.

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- The well-monotone quasi-uniformity on a strictly zero-dimensional biframe $\mathcal{L}$ is generated by paircovers of the form

$$C_A = \bigcap_{a \in A} ({\downarrow}(a, 1) \cup {\downarrow}(1, a^{\mathsf{c}}))$$

where A is a *well-ordered cover* of $\mathcal{L}_1$.

- Then $C_A = \bigcup_{b \in A} {\downarrow}(b, (b^-)^{\mathsf{c}})$, where $b^- = \bigvee\{a \in A \mid a < b\}$.

Bicompleteness in the well-monotone quasi-uniformity

Theorem

A strictly zero-dimensional biframe $\mathcal{L}$ is bicomplete in the well-monotone quasi-uniformity iff it is a congruence biframe.

Furthermore, the underlying biframe of the bicompletion with respect to the well-monotone quasi-uniformity is the congruential coreflection.

Corollary (Plewe)

Congruence frames are ultraparacompact — i.e. every open cover admits a refinement into a partition.

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In summary

T_0 **spaces** $\simeq$	**Skula biframes**	**Strictly 0d biframes**
Sober spaces	*Sober Skula biframes*	*Congruence biframes*
Irreducible closed sets are closures of discrete subspaces	Prime closed elements are clarifiable	All closed elements are clarifiable
Universally Skula-closed	Universally closed	Universally closed
Bicomplete in the well-monotone quasi-uniformity	*Cauchy bicomplete* in the well-monotone quasi-uniformity	Bicomplete in the well-monotone quasi-uniformity
$\mathrm{sob} \cong \Sigma_1(\mathbb{C}\mathfrak{F})\mathrm{Sk}$	$\mathrm{Sk}\Sigma\mathfrak{F} \cong (\mathrm{Sk}\Sigma_1)(\mathbb{C}\mathfrak{F})$	$\mathbb{C}\mathfrak{F}$

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