Parts of biframes and a categorical approach to **BiFrm**

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WorkALT
in honour of Aleš Pultr, on the occasion of his 80th birthday

Universidad del País Vasco / Euskal Herriko Unibertsitatea

[1] Joint work with Andrew Moshier and Joanne Walters-Wayland

Parts of pointfree spaces: Sublocales

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Extremal epimorphisms

An extremal epimorphism e in a category C is an epimorphism such that if

$$e = m \circ g \text{ where } m \text{ is a monomorphism} \implies m \text{ is an isomorphism.}$$

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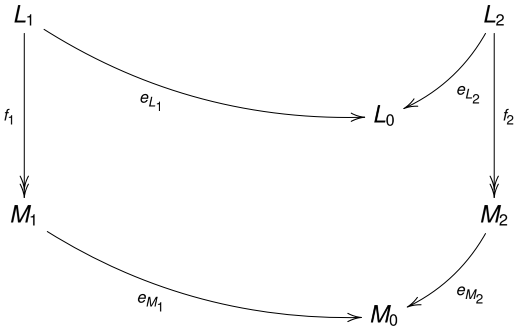

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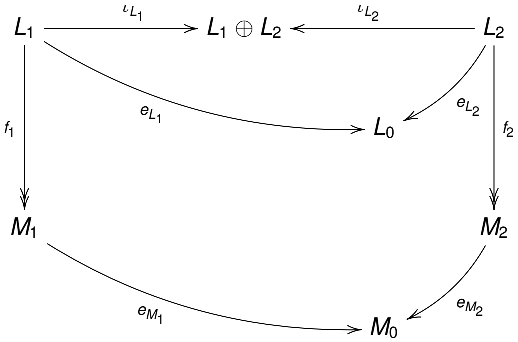

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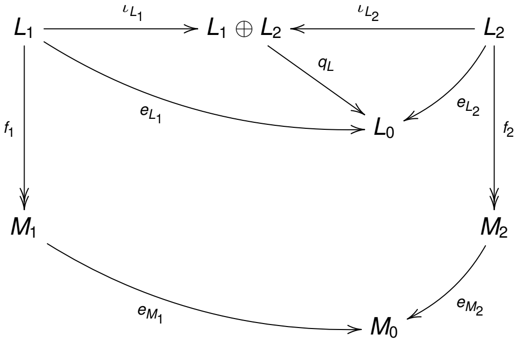

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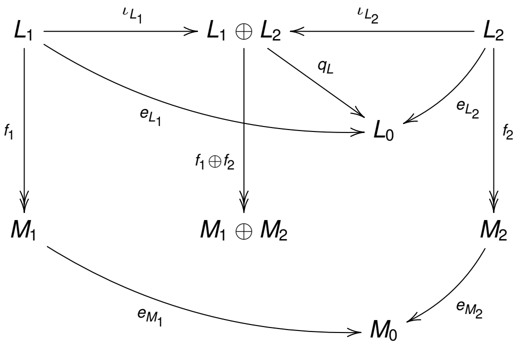

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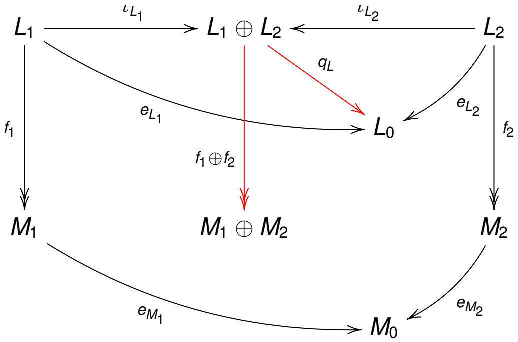

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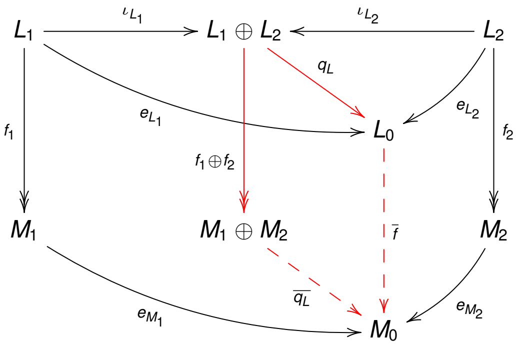

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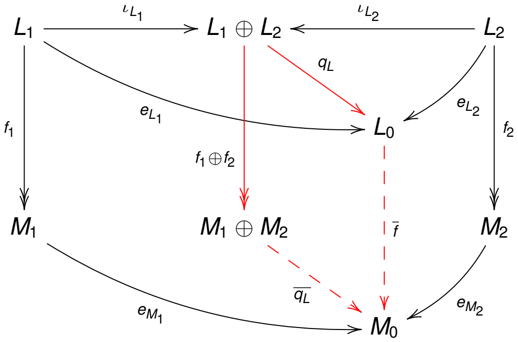

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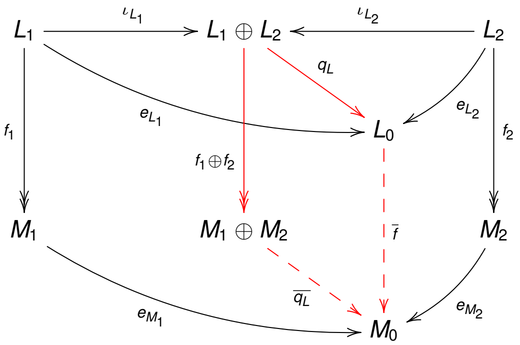

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Let $f\colon L \to S$ be an extremal epimorphism in **BiFrm**. *Then $[f_0] = \mathcal{S}(e_{L_1})[f_1] \wedge \mathcal{S}(e_{L_2})[f_2]$.*

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Proposition

Let L be a biframe. For any $[f_1] \in \mathcal{S}(L_1)$ and $[f_2] \in \mathcal{S}(L_2)$ there exists a an extremal epimorphism $f \colon L \to M$ such that $[f_0] = \mathcal{S}(e_{L_1})[f_1] \wedge \mathcal{S}(e_{L_2})[f_2]$.

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Let $f_0\colon L_0 \to M_0 \in \mathbf{Frm}$ and extremal epi. One has a biframe homomorphism

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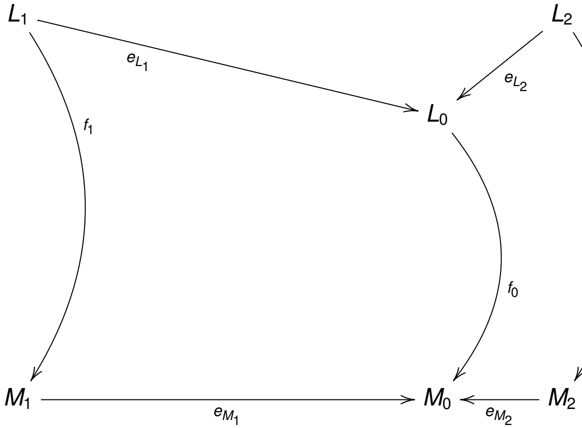

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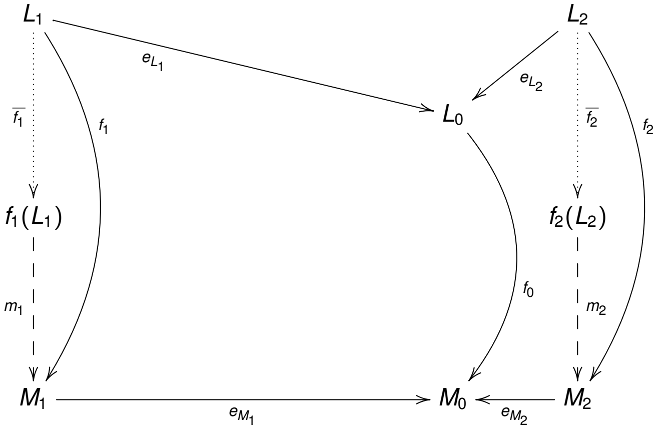

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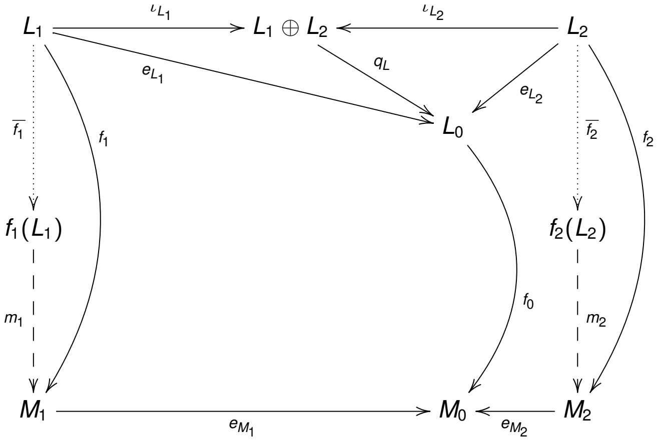

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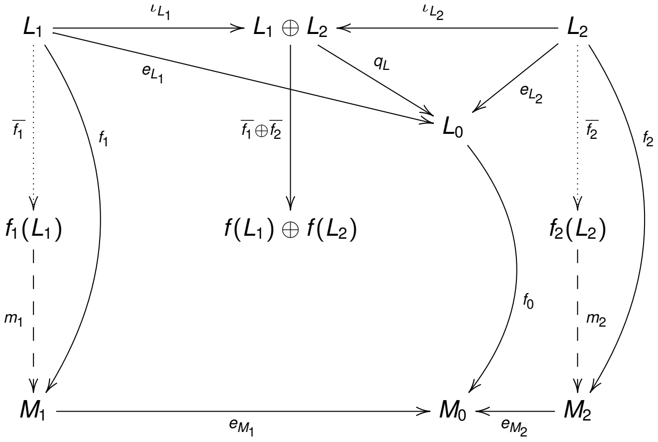

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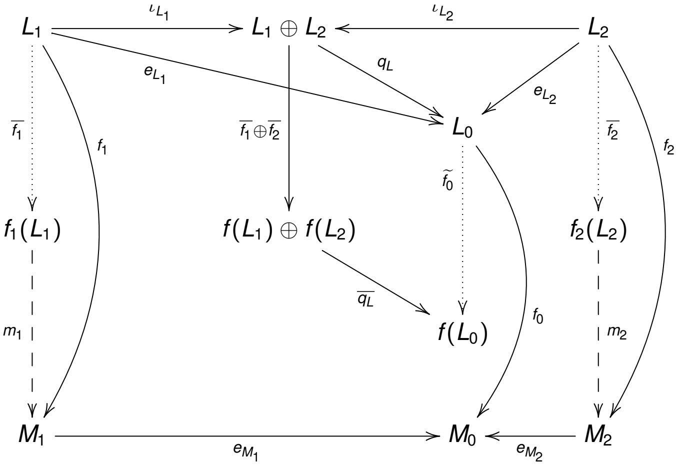

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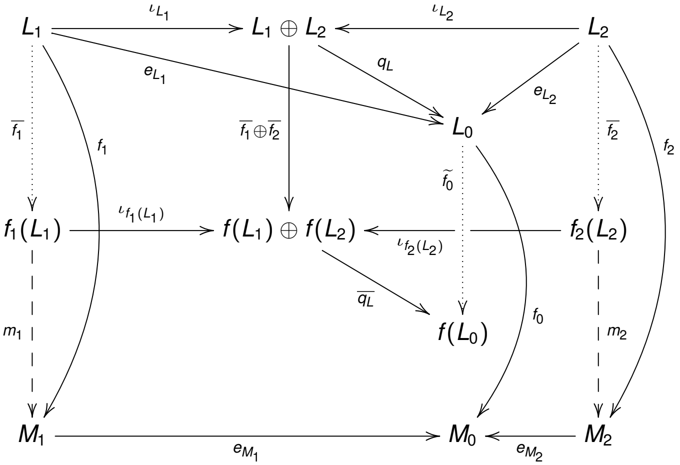

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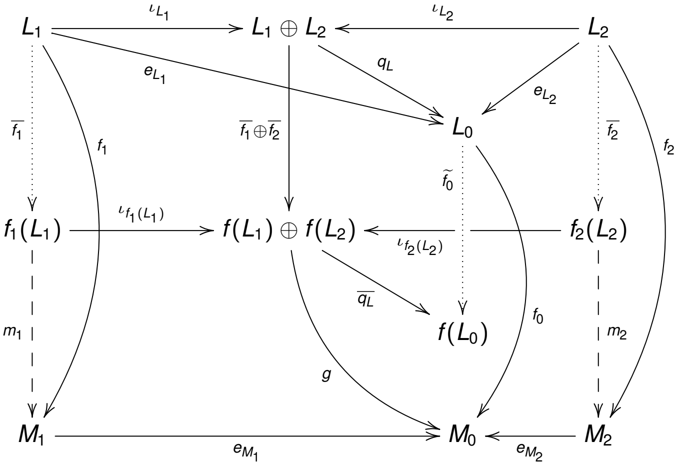

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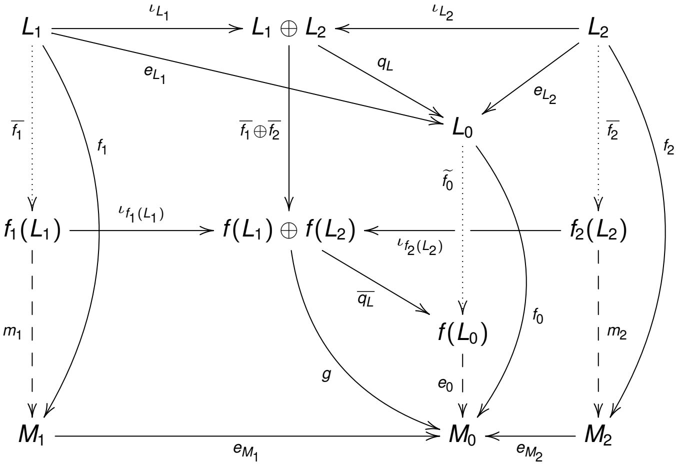

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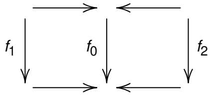

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The assignment $f_0 \mapsto \overline{f_0}$ determines a closure operator on the poset of quotients of C.

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