The meet-semilattice congruence lattice of a frame

John Frith* and Anneliese Schauerte

University of Cape Town

27 September 2018

Basics

Throughout this talk L will denote a frame, the top element is denoted by 1 the bottom element is denoted by 0.

Definition

A *meet-semilattice congruence* θ on L is an equivalence relation on L which also satisfies $(x, y), (z, w) \in \theta \Rightarrow (x \wedge z, y \wedge w) \in \theta$.

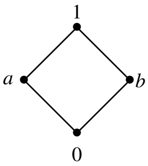
We present some well-known facts for the sake of completeness:

- The collection of all meet-semilattice congruences on L, $\mathrm{Con}_{\mathrm{Msl}}(L)$, forms a partially ordered set under inclusion.
- The intersection of meet-semilattice congruences remains a meet-semilattice congruence, so meet is given by intersection.
- $\mathrm{Con}_{\mathrm{Msl}}(L)$ is a complete lattice. The top element, which we denote by ∇, is $L \times L$; the bottom element, which we denote by $\triangle$, is $\{(x, x) : x \in L\}$.

An example

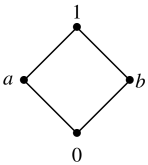
An example

L:

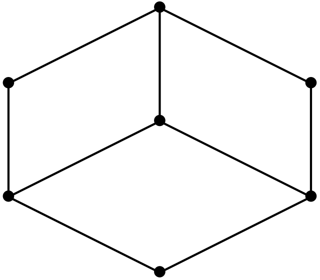


An example

L:



$\mathrm{Con}_{\mathrm{Msl}}(L)$:



Finite joins

There is an explicit characterization of finite joins, which we need, given as follows:
Suppose that θ, ϕ are meet-semilattice congruences on L.

- We say that elements x and y of L are $\theta - \phi$-*linked* if there is a sequence of elements $x = s_0, s_1, s_2 \ldots, s_n = y$ of L such that, for any $i \in \{0, 1, 2, \ldots, n-1\}$ either $(s_i, s_{i+1}) \in \theta$ or $(s_i, s_{i+1}) \in \phi$.
- We define $\theta * \phi = \{(x, y) : x \text{ and } y \text{ are } \theta - \phi\text{-linked}\}$
- For $\theta, \phi \in \mathrm{Con}_{\mathrm{Msl}}(L)$, $\theta \vee \phi = \theta * \phi$. Well known, we think. This extends to any finite join.

$\mathrm{Con}_{\mathrm{Msl}}(L)$

- The join of an updirected family of meet-semilattice congruences is just its union.
- An arbitrary join, $\bigvee_I \theta_i$, is calculated by taking the union of all finite joins, since these form an updirected collection.
- As a result, the lattice $\mathrm{Con}_{\mathrm{Msl}}(L)$ is compact.

For the sake of completeness, we recall the definition of a frame congruence:

Definition

A *frame congruence* θ on L is an equivalence relation on L which also satisfies

- $(x, y), (z, w) \in \theta$ implies $(x \wedge z, y \wedge w) \in \theta$.
- $(x_i, y_i) \in \theta$ for all $i \in I$ implies $(\bigvee_I x_i, \bigvee_I y_i) \in \theta$.

- The collection of all frame congruences on a frame L will be denoted by $\mathrm{Con}_{\mathrm{Frm}}(L)$.
- It is a frame. (But the description of join given above does not apply.)

A structure theorem for $\mathrm{Con}_{\mathrm{Msl}}(L)$.

Definition

For $a, b \in L$ we denote by

- $\blacktriangledown_a$ the meet-semilattice congruence generated by the singleton $\{(0, a)\}$
- Δ_b the meet-semilattice congruence generated by the singleton $\{(b, 1)\}$
- θ_{ab} the meet-semilattice congruence generated by the singleton $\{(a, b)\}$.
- $\nabla_a = \{(x, y) \in L \times L : x \vee a = y \vee a\}$.

It is possible to describe $\blacktriangledown_a$ and Δ_b explicitly as follows:

Lemma

For $a, b \in L$

1. $\blacktriangledown_a = \triangle \cup \{(s,t) \in L \times L : s,t \leq a\}$
2. $\Delta_b = \{(x,y) \in L \times L : x \wedge b = y \wedge b\}$.

Some properties of $\blacktriangledown_a$, etc.

Some properties of $\blacktriangledown_a$, etc.

Lemma

Let L be a frame, $a, b \in L$, $\{a_i\}_{i \in I} \subseteq L$. In $\mathrm{Con}_{\mathrm{Msl}}(L)$ we have:

1. (a) $\bigwedge_I \blacktriangledown_{a_i} = \blacktriangledown_{\bigwedge a_i}$.
   (b) $\blacktriangledown_a \vee \blacktriangledown_b$ need not coincide with $\blacktriangledown_{a \vee b}$.
   (c) $\blacktriangledown_0 = \triangle$ and $\blacktriangledown_1 = \nabla$.
2. (a) $\nabla_a \wedge \nabla_b = \nabla_{a \wedge b}$.
   (b) $\nabla_a \vee \nabla_b = \nabla_{a \vee b}$; this does not generalize to arbitrary joins.
   (c) $\nabla_0 = \triangle$ and $\nabla_1 = \nabla$.
3. (a) $\bigwedge \Delta_{a_i} = \Delta_{\bigvee a_i}$.
   (b) $\Delta_a \vee \Delta_b = \Delta_{a \wedge b}$; this does not generalize to arbitrary joins.
   (c) $\Delta_0 = \nabla$ and $\Delta_1 = \triangle$.

Towards some structure

Lemma

$\theta_{ab} = (\blacktriangledown_a \wedge \Delta_b) * (\blacktriangledown_b \wedge \Delta_a).$

Towards some structure

Lemma

$\theta_{ab} = (\blacktriangledown_a \wedge \Delta_b) * (\blacktriangledown_b \wedge \Delta_a)$.

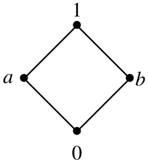
Theorem (Structure Theorem)

For any meet-semilattice congruence θ we have

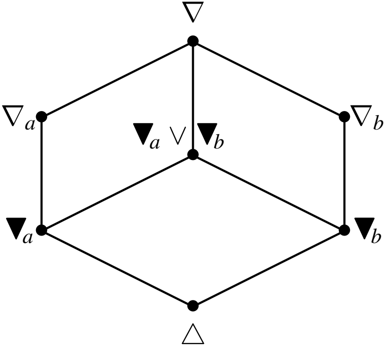
$$\theta = \bigvee\{(\blacktriangledown_c \wedge \Delta_d) * (\blacktriangledown_d \wedge \Delta_c) : (c, d) \in \theta\}.$$

Examples and counterexamples

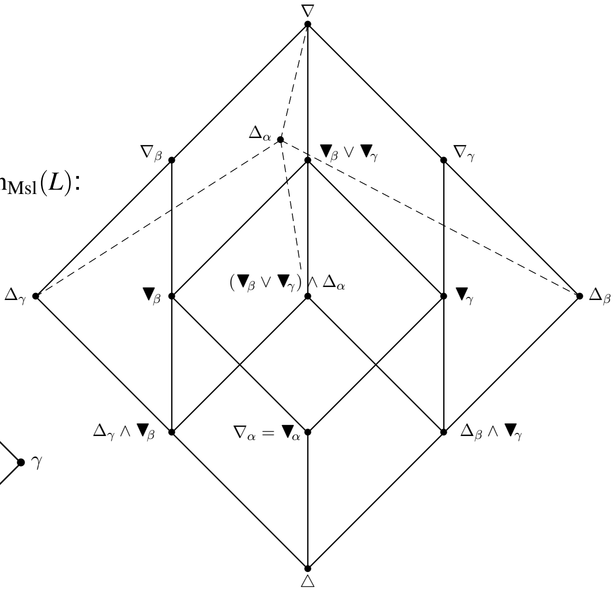
L:



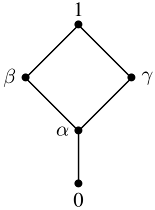
$\mathrm{Con}_{\mathrm{Msl}}(L)$:



$\mathrm{Con}_{\mathrm{Msl}}(L)$:



L:



Theorem

In the case that L is a linear frame we claim that $\mathrm{Con}_{\mathrm{Msl}}(L)$ is indeed a frame but that, in general, $\mathrm{Con}_{\mathrm{Msl}}(L) \neq \mathrm{Con}_{\mathrm{Frm}}(L)$. (See Example below.)

The proof that we found relies on the "Structure Theorem" and follows a similar route to a proof that the congruence lattice of a frame is again a frame.

Theorem

In the case that L is a linear frame we claim that $\mathrm{Con}_{\mathrm{Msl}}(L)$ is indeed a frame but that, in general, $\mathrm{Con}_{\mathrm{Msl}}(L) \neq \mathrm{Con}_{\mathrm{Frm}}(L)$. (See Example below.)

The proof that we found relies on the "Structure Theorem" and follows a similar route to a proof that the congruence lattice of a frame is again a frame.
Papert has this result, but it is proved differently.

Theorem

In the case that L is a linear frame we claim that $\mathrm{Con}_{\mathrm{Msl}}(L)$ is indeed a frame but that, in general, $\mathrm{Con}_{\mathrm{Msl}}(L) \neq \mathrm{Con}_{\mathrm{Frm}}(L)$. (See Example below.)

The proof that we found relies on the "Structure Theorem" and follows a similar route to a proof that the congruence lattice of a frame is again a frame.
Papert has this result, but it is proved differently.

Example

As a special case of a linear frame L we take $L = \mathbb{N} \cup \{\top\}$ where $\mathbb{N}$ denotes the positive integers with their usual order and $n \leq \top$ for all $n \in \mathbb{N}$. This is clearly a case where L is a linear frame. One can see that $\mathrm{Con}_{\mathrm{Msl}}(L) \neq \mathrm{Con}_{\mathrm{Frm}}(L)$.

Lemma

If the frame L has at least two incomparable elements, then $\mathrm{Con}_{\mathrm{Msl}}(L)$ is not a distributive lattice.

Lemma

If the frame L has at least two incomparable elements, then $\mathrm{Con}_{\mathrm{Msl}}(L)$ is not a distributive lattice.

PROOF. The proof is modelled on the case where L is the 4 element Boolean algebra. □

Complements in $\mathrm{Con}_{\mathrm{Msl}}(L)$?

We define complements using the usual equations as follows:

Definition

Let M be a bounded lattice; for $a, b \in M$ we say that a is a *complement* of b if $a \vee b = 1$ and $a \wedge b = 0$.

We emphasize that we are using this definition in a possibly non-distributive lattice and so no implication of uniqueness is intended.

Lemma

Let L be a frame, $a \in L$. In $\mathrm{Con}_{\mathrm{Msl}}(L)$, the element $\blacktriangledown_a$ has a unique complement, namely, Δ_a.

Complements in $\mathrm{Con}_{\mathrm{Msl}}(L)$?

We define complements using the usual equations as follows:

Definition

Let M be a bounded lattice; for $a, b \in M$ we say that a is a *complement* of b if $a \vee b = 1$ and $a \wedge b = 0$.

We emphasize that we are using this definition in a possibly non-distributive lattice and so no implication of uniqueness is intended.

Lemma

Let L be a frame, $a \in L$. In $\mathrm{Con}_{\mathrm{Msl}}(L)$, the element $\blacktriangledown_a$ has a unique complement, namely, Δ_a.

Lemma

Let L be a frame, $a \in L$. In $\mathrm{Con}_{\mathrm{Msl}}(L)$ if θ is a complement of Δ_a, then $\blacktriangledown_a \subseteq \theta \subseteq \nabla_a$.

Lemma

Let L be a frame, $a, b \in L$. In $\mathrm{Con}_{\mathrm{Msl}}(L)$, every element of the form $\blacktriangledown_a \wedge \Delta_b$ is complemented.

PROOF. $\blacktriangledown_b * \Delta_a$ is a complement of $\blacktriangledown_a \wedge \Delta_b$ □

- We now see that all elements of $\mathrm{Con}_{\mathrm{Msl}}(L)$ arise as joins of complemented elements (using the Structure Theorem). In this sense one may think of any such lattice as being "zero-dimensional." We see that every meet-semilattice congruence lattice is compact and zero-dimensional.
- However, not every compact zero-dimensional lattice (in this sense) is a meet-semilattice congruence lattice of some frame.

Lemma

Let L be a frame, $a, b \in L$. In $\mathrm{Con}_{\mathrm{Msl}}(L)$, every element of the form $\blacktriangledown_a \wedge \Delta_b$ is complemented.

PROOF. $\blacktriangledown_b * \Delta_a$ is a complement of $\blacktriangledown_a \wedge \Delta_b$ □

- We now see that all elements of $\mathrm{Con}_{\mathrm{Msl}}(L)$ arise as joins of complemented elements (using the Structure Theorem). In this sense one may think of any such lattice as being "zero-dimensional." We see that every meet-semilattice congruence lattice is compact and zero-dimensional.
- However, not every compact zero-dimensional lattice (in this sense) is a meet-semilattice congruence lattice of some frame.
- Papert proves that $\mathrm{Con}_{\mathrm{Msl}}(L)$ is always pseudo-complemented.

Functoriality issues

Throughout: $f : L \to M$ is a frame map between frames.

Definition

$(f \times f)^{-1} : \mathrm{Con}_{\mathrm{Msl}}(M) \to \mathrm{Con}_{\mathrm{Msl}}(L)$ has a left adjoint from $\mathrm{Con}_{\mathrm{Msl}}(L)$ to $\mathrm{Con}_{\mathrm{Msl}}(M)$ which we denote by $\widetilde{f} : \mathrm{Con}_{\mathrm{Msl}}(L) \to \mathrm{Con}_{\mathrm{Msl}}(M)$.

- For any $\theta \in \mathrm{Con}_{\mathrm{Msl}}(L)$ we have that $\widetilde{f}(\theta) = \langle (f \times f)[\theta] \rangle$.
- $\widetilde{f}$ preserves arbitrary joins, since it is a left adjoint.
- It therefore preserves the bottom element.

Functoriality issues

Throughout: $f : L \to M$ is a frame map between frames.

Definition

$(f \times f)^{-1} : \mathrm{Con}_{\mathrm{Msl}}(M) \to \mathrm{Con}_{\mathrm{Msl}}(L)$ has a left adjoint from $\mathrm{Con}_{\mathrm{Msl}}(L)$ to $\mathrm{Con}_{\mathrm{Msl}}(M)$ which we denote by $\mathcal{f} : \mathrm{Con}_{\mathrm{Msl}}(L) \to \mathrm{Con}_{\mathrm{Msl}}(M)$.

- For any $\theta \in \mathrm{Con}_{\mathrm{Msl}}(L)$ we have that $\mathcal{f}(\theta) = \langle (f \times f)[\theta] \rangle$.
- $\mathcal{f}$ preserves arbitrary joins, since it is a left adjoint.
- It therefore preserves the bottom element.
- It also preserves the top element.

Functoriality issues

Throughout: $f : L \to M$ is a frame map between frames.

Definition

$(f \times f)^{-1} : \mathrm{Con}_{\mathrm{Msl}}(M) \to \mathrm{Con}_{\mathrm{Msl}}(L)$ has a left adjoint from $\mathrm{Con}_{\mathrm{Msl}}(L)$ to $\mathrm{Con}_{\mathrm{Msl}}(M)$ which we denote by $\mathcal{F} : \mathrm{Con}_{\mathrm{Msl}}(L) \to \mathrm{Con}_{\mathrm{Msl}}(M)$.

- For any $\theta \in \mathrm{Con}_{\mathrm{Msl}}(L)$ we have that $\mathcal{F}(\theta) = \langle (f \times f)[\theta] \rangle$.
- $\mathcal{F}$ preserves arbitrary joins, since it is a left adjoint.
- It therefore preserves the bottom element.
- It also preserves the top element.
- $\mathcal{F}(\theta_{ab}) = \theta_{f(a)f(b)}, \mathcal{F}(\blacktriangledown_a) = \blacktriangledown_{f(a)}, \mathcal{F}(\Delta_a) = \Delta_{f(a)}$.

It is now clear that the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\blacktriangledown_L} & \mathrm{Con}_{\mathrm{Msl}}(L) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{F}} \\ M & \xrightarrow[\blacktriangledown_M]{} & \mathrm{Con}_{\mathrm{Msl}}(M) \end{array}$$

We note that in the diagram above, the horizontal maps preserve arbitrary meets, whereas the vertical maps preserve arbitrary joins.

$\mathrm{Con}_{\mathrm{Msl}}(L)$ and $\mathrm{Con}_{\mathrm{Frm}}(L)$

The following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Con}_{\mathrm{Frm}}(L) & \xrightarrow{\;\widehat{f}\;} & \mathrm{Con}_{\mathrm{Frm}}(M) \\
{\scriptstyle b_L}\uparrow & & \uparrow{\scriptstyle b_M} \\
\mathrm{Con}_{\mathrm{Msl}}(L) & \xrightarrow[\;\widetilde{f}\;]{} & \mathrm{Con}_{\mathrm{Msl}}(M)
\end{array}$$

We note that all maps in the diagram above preserve (at least) arbitrary joins, top and bottom elements.

The following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Con}_{\mathrm{Frm}}(L) & \xrightarrow{\quad \widehat{f} \quad} & \mathrm{Con}_{\mathrm{Frm}}(M) \\
{\scriptstyle i_L}\big\downarrow & & \big\uparrow{\scriptstyle b_M} \\
\mathrm{Con}_{\mathrm{Msl}}(L) & \xrightarrow[\quad \mathcal{F} \quad]{} & \mathrm{Con}_{\mathrm{Msl}}(M)
\end{array}
$$

The following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Con}_{\mathrm{Frm}}(L) & \xrightarrow{\;\widehat{f}\;} & \mathrm{Con}_{\mathrm{Frm}}(M) \\
{\scriptstyle i_L}\big\downarrow & & \big\uparrow{\scriptstyle b_M} \\
\mathrm{Con}_{\mathrm{Msl}}(L) & \xrightarrow[\;\tilde{f}\;]{} & \mathrm{Con}_{\mathrm{Msl}}(M)
\end{array}$$

Muito obrigado a todos vocês e especialmente a Aleš